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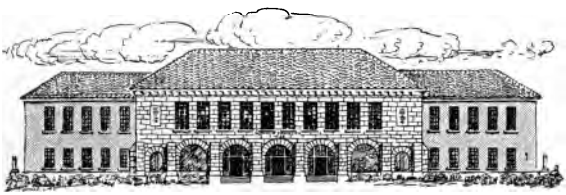

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PLANE AND SOLID

GEOMETRY

BY

SETH T. STEWART

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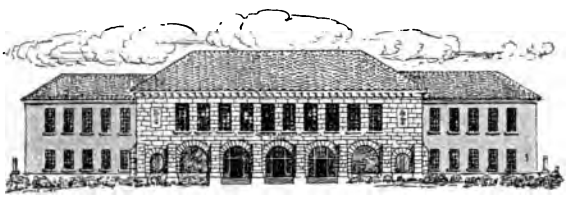

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PLANE AND SOLID

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PREFACE.

GEOMETRY is the science best adapted to the development of the faculties of the student in deductive reasoning. Among the invaluable results of successful work in this subject are the habit of forming distinct conceptions and the power of continuity in thought and speech. The knowledge, moreover, to be acquired by its study is of the highest importance, not only in the pursuit of allied sciences, but also in many useful arts and industries.

A prominent feature of this text-book is its strict adherence to the principle of association. Each book treats of one subject, and each section treats of one subdivision of the subject. The student will thus not only acquire more distinct and more general concepts, but his memory will have the full benefit of the operation of the laws of association of thought.

Another and perhaps the leading feature of the book is the system of graded exercises. A body of work is here presented in a form rendered possible only by the grouping of propositions. At the end of each section miscellaneous exercises, thoroughly assorted and graded, are presented in such order that their successive solution causes a constant growth in the power of thought, analytic and synthetic. A synopsis of each book precedes the book itself, to encourage students to work independently of the demonstrations given

in the text. The entire volume thus becomes one complete set of graded exercises.

The miscellaneous exercises in Books II., III., and IV. are not only assorted and graded in kind, but they are graded into three degrees of difficulty. The second grade of exercises may be reserved for review, and the third grade for subsequent review, or examination.

Teachers desiring a shorter course may omit not only many of the exercises, of which there are more than nine hundred, but also some of the more difficult sections of the several books. The books themselves, moreover, should be treated as sets of exercises, and it will be found that several propositions, in some cases, may be assigned for a single lesson. To a student, thus, that has mastered Book II., the propositions of Book III. will prove to be easy exercises, of which several may form but one lesson.

The subject of proportion is developed in Book V., and this being the basis of mensuration, it is immediately followed by two books on the application of proportion to the mensuration of plane figures.

In the treatment of each part of the work, the inductive method, now in general use by successful teachers in more elementary studies, has been employed. While the approaches to the subject are thus rendered more agreeable, the author has been conservative in retaining, so far as possible, the usual phraseology of propositions and a wholesome rigor in demonstration.

At the end of several books, general scholia are given, not as an essential part of the volume, but as matter of interest to the more advanced student.

In the presentation of propositions, the several parts are distinctly named: Theorem or Problem, Statement, Con-

struction, Demonstration, and Conclusion. The student thus becomes familiar with the proper sequence, and loses no time in finding any desired part of a proposition. The important language lessons, moreover, involved in the study of Geometry are not marred by disjointed sentences and irregular typography; but the entire text, with the exception of the theorems or problems, is printed in ordinary type.

The attention of teachers is called to the Table of Contents and to the headlines—the one much the same as the other. The student may thus readily find any desired subject, and know at a glance to which part of the book he may have opened.

It will be noted that a definite plan is followed with respect to diagrams. Lines mentioned in the *statement* are solid lines, unless planes intervene; while additional lines, mentioned in the *construction*, are dotted lines. The solid lines, moreover, represent that which is given, the more important lines being heavier than the less important.

In Solid Geometry, the diagrams are drawn with full regard to perspective. Throughout the volume, the diagrams and demonstrations are invariably in full view of each other.

As different purposes are served in teaching Geometry and Algebra—two distinct forms of mathematical thought—the author, so far as possible, has pursued the methods peculiar to Geometry. A large number of numerical exercises will be found distributed throughout the volume.

Criticisms and suggestions will be gratefully received.

SETH T. STEWART.

BROOKLYN, *June* 23, 1891.

CONTENTS.

INTRODUCTION.

PAGE

RULE AND COMPASS, ... 1
DEFINITIONS,* 4
AXIOMS, ... 6
POSTULATES, ... 6

PLANE GEOMETRY.

BOOK I.—STRAIGHT LINES AND ANGLES.

SYNOPSIS, ... 7
SECTION I.—SUPERPOSITION, ... 10
Miscellaneous Exercises,† 15
SECTION II.—ANGLES, ... 16
SECTION III.—PERPENDICULAR LINES, ... 23
Miscellaneous Exercises, ... 30
SECTION IV.—PARALLEL LINES, ... 31
SECTION V.—OBLIQUE LINES, ... 40
Miscellaneous Exercises, ... 47
GENERAL SCHOLIA—THEORY OF LOCI, SCIENCE OF GEOMETRY, . 49

BOOK II.—TRIANGLES.

SYNOPSIS, ... 53
SECTION I.—EQUAL TRIANGLES, ... 56
Miscellaneous Exercises, ... 64
SECTION II.—EQUAL PARTS, ... 66
Miscellaneous Exercises, ... 72

* The main body of the definitions is distributed throughout the volume, preceding, in each case, the appropriate sections.

† Many exercises are found in different sections, in addition to those referred to in the Table of Contents at the end of the respective sections.

PAGE

SECTION III.—UNEQUAL PARTS, ... 75
Miscellaneous Exercises, ... 81
SECTION IV.—ANGLES, ... 82
Miscellaneous Exercises, ... 87
SECTION V.—CONCURRENT LINES, ... 90
Miscellaneous Exercises, ... 100
GENERAL SCHOLIA—INTERSECTION OF LOCI, SYMMETRY, . . 101

BOOK III.—QUADRILATERALS.

SYNOPSIS, ... 102
SECTION I.—EQUAL SIDES, ... 105
Miscellaneous Exercises, ... 111
SECTION II.—ANGLES, ... 112
Miscellaneous Exercises, ... 117
SECTION III.—DIAGONALS AND DIAMETERS, ... 117
Miscellaneous Exercises, ... 123
SECTION IV.—EQUAL AREAS, ... 124
Miscellaneous Exercises, ... 129
SECTION V.—COMPARISON OF AREAS, ... 130
Miscellaneous Exercises, ... 140
GENERAL SCHOLIA—MAXIMA AND MINIMA, THEORY OF LIMITS, . 148

BOOK IV.—CIRCLES.

SYNOPSIS, ... 151
SECTION I.—EQUAL CHORDS AND ARCS, ... 154
Miscellaneous Exercises, ... 160
SECTION II.—UNEQUAL CHORDS AND ARCS, ... 161
Miscellaneous Exercises, ... 166
SECTION III.—CHORDS AND PERPENDICULARS, ... 167
Miscellaneous Exercises, ... 172
SECTION IV.—ANGLES AND ARCS, ... 175
Miscellaneous Exercises, ... 181
SECTION V.—TWO CIRCLES, ... 184
Miscellaneous Exercises, ... 190
GENERAL SCHOLIA—CONCYCLIC POINTS, INSCRIBED AND ESCRIBED CIRCLES, THE TRIANGLE AND NINE OF ITS CIRCLES, THE NINE-POINTS CIRCLE, ... 196

BOOK V.—PROPORTION.

PAGE

SYNOPSIS, ... 199
SECTION I.—EQUAL RATIOS, ... 201
SECTION II.—EQUAL PRODUCTS, ... 206
SECTION III.—SEVERAL PROPORTIONS, ... 208
SECTION IV.—CONTINUED PROPORTIONS, ... 211

BOOK VI.—MENSURATION: PLANE FIGURES.

SYNOPSIS, ... 214
SECTION I.—ANGLES, ... 217
SECTION II.—TRIANGLES. 223
Miscellaneous Exercises, ... 229
SECTION III.—QUADRILATERALS. 230
Miscellaneous Exercises, ... 236
SECTION IV.—REGULAR POLYGONS, ... 237
Miscellaneous Exercises, ... 245
SECTION V.—CIRCLES, ... 246
Miscellaneous Exercises, ... 250

BOOK VII.—MENSURATION: SIMILAR PLANE FIGURES.

SYNOPSIS, ... 255
SECTION I.—TRIANGLES, ... 258
Miscellaneous Exercises, ... 265
SECTION II.—PARTS OF TRIANGLES, ... 266
Miscellaneous Exercises, ... 271
SECTION III.—POLYGONS, ... 274
Miscellaneous Exercises, ... 279
SECTION IV.—CIRCLES, ... 280
Miscellaneous Exercises, ... 285
SECTION V.—PROPORTIONAL AREAS, ... 290
Miscellaneous Exercises, ... 297

SOLID GEOMETRY.

BOOK VIII.—PLANES.

PAGE

SYNOPSIS, ... 301
SECTION I.—ONE PLANE, ... 303
Miscellaneous Exercises, ... 310
SECTION II.—TWO PLANES, ... 311
Miscellaneous Exercises, ... 317
SECTION III.—SEVERAL PLANES, ... 318
Miscellaneous Exercises, ... 323

BOOK IX.—SOLIDS WITH PLANE SURFACES.

SYNOPSIS, ... 325
SECTION I.—SECTIONS, ... 327
Miscellaneous Exercises, ... 333
SECTION II.—SURFACES, ... 335
Miscellaneous Exercises, ... 337, 341
SECTION I.—EQUIVALENT SOLIDS, ... 342
Miscellaneous Exercises, ... 347
SECTION IV PROPORTIONAL SOLIDS, ... 348
Miscellaneous Exercises, ... 351
SECTION V.—VOLUMES, ... 353
Miscellaneous Exercises, ... 358

BOOK X.—SOLIDS WITH CURVED SURFACES.

SYNOPSIS, ... 360
SECTION I —SECTIONS, ... 362
Miscellaneous Exercises, ... 368, 370
SECTION II.—SURFACES, ... 371
Miscellaneous Exercises, ... 378
SECTION III. VOLUMES, ... 379
Miscellaneous Exercises, ... 383, 386
SECTION IV —POLARITY AND SYMMETRY, ... 387
Miscellaneous Exercises, ... 396
SECTION V SPHERICAL TRIANGLES, ... 397
Miscellaneous Exercises, ... 402

NOTE.

ABBREVIATIONS may be used in written recitation, by the student, as follows :

⊥, perpendicular ;	∠, angle ;
⊥s, perpendiculars ;	∠s, angles ;
∥, parallel ;	∟, right angle ;
∥s, parallels ;	∟s, right angles.

△, triangle ;	○, circle ;
△s, triangles ;	○s, circles ;
▭, parallelogram ;	⌒, arc ;
▭s, parallelograms ;	⌒s, arcs.

+, plus ;	=, equals, *or*, is equal to ;
−, minus ;	≎, is equivalent to ;
×, multiplied by ;	>, is greater than ;
÷, divided by ;	<, is less than ;
∵, hence ;	∴, therefore.

Quod erat demonstrandum (*Q. E. D.*), which was to be proved.

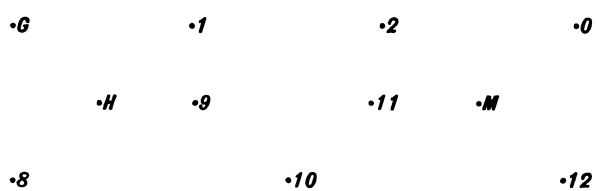
Quod erat faciendum (*Q. E. F.*), which was to be done.

ELEMENTS OF GEOMETRY.

INTRODUCTION.

RULE AND COMPASS.*

1. Exercise I. *With points:* 1. Place several dots on the blackboard, paper, or slate, representing as many points.



1. Name the points, as the point *1* or the point *2*, the point *M* or the point *0*, etc. 2. Where are points *G*, *H*, *9*, and *12*? 3. What has every point? Has it length, breadth, thickness, or only position? The dots have size, but the points have neither length, breadth, nor thickness. They have position only.

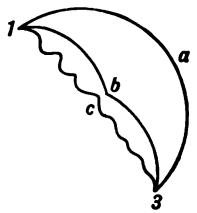
2. Indicate two points on the writing surface, as the points *1* and *2*.

1. As each point has position, so any two points have relative position. 2. With one end of the rule on the point *8*, direct the other end toward the point *1 1*. 3. With one end of the rule on the point *1 1*, direct the other end toward the point *8*. 4. What name shall be given to the relative position of two points? What is meant by direction?

---

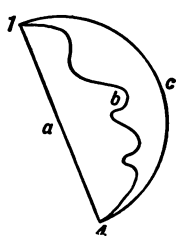
*The exercises under Rule and Compass, preceding sections of the first few books, are intended to develop ideas subsequently embodied in definitions and, also, to enable the student to make correct constructions of required figures.

2. Exercise II. *With lines:* 1. Between any two points draw several lines, as between the points *1* and *3*.



1. What has each line besides position and direction? 2. Are the lines of equal length? 3. What is, then, the peculiar property of lines? Length. 4. The marks have breadth, but the lines are supposed to have no breadth.

2. Indicate any two points, as the points *1* and *4*, and, placing the ruler so that the two points are against the same edge, draw a line from the one point to the other: viz., the line *1 a 4*; draw also other lines between *1* and *4*, as *1 b 4*, and *1 c 4*.



1. What kinds of lines are the different lines thus drawn? 2. Which of the lines is made without change of direction? 3. What kind of a line is it? 4. Which line is a straight line? 5. Which line might be called a crooked line, and which a curved line? 6. Which line seems to be made with irregular changes of direction? 7. Which line seems to be made with regular changes of direction?

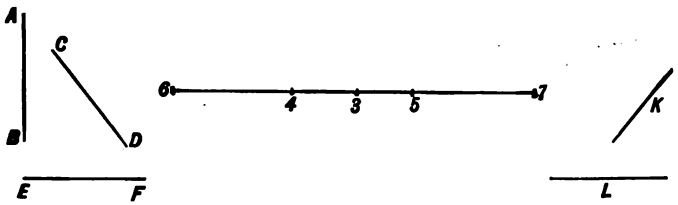
3. Draw a straight line between any two points.

1. Draw, if possible, a shorter line between the same two points. 2. Which line, then, is the shortest line that can be drawn between any two points?

4. From one point to another, or from one point in the direction of another point, draw a straight line.

1. Draw, if possible, another straight line, either (*a*) between the same two points, or (*b*) from the one point in the direction of the other point. 2. How many straight lines can be drawn between the same two points? 3. How many straight lines can be drawn from one point in the direction of another point? 4. What will determine, then, the directions of a straight line?

5. By using the rule, and also by using the compass or dividers, compare the lengths of any two straight lines.

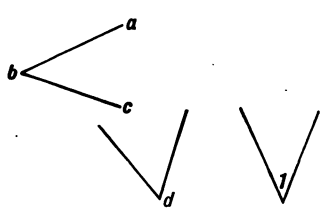


1. Is the line *K* or the line *L*, the longer, by the compass and by the rule? 2. How long is the line *K*? How long is the line *L*? 3. Show by the dividers that two of the lines on this page are equal. 4. Show by the rule that two other lines are equal.

6. By the dividers, lay off one line on another, or take a part of one line equal to another line.

1. From *6* lay off *A B* on the line *6 7*. What part of the line does it cover? 2. Lay off the same line on *6 7* from *7*. 3. In *4 7* take any point *3*, and cut off a part of *3 7* equal to *3 4*.

3. Exercise III. *With angles:* 1. Draw a straight line from any point in any direction, and from the same point draw a straight line in any other direction.



1. What name shall be given to the difference of direction between two straight lines that meet? 2. What is an angle? 3. Where is the angle *a b c*? *c b a*? *d*? *1*? 4. Make a number of angles and indicate their names in different ways. 5. Point to different angles in the structure of the recitation room, furniture, etc.

2. From any given point, draw a number of straight lines.

1. Indicate all the angles thus made at the given point. 2. How many angles can be made at a given point?

DEFINITIONS.

4. A **point** is that which has position only.

5. **Direction** is relative position of points.

From one point toward a second point is one direction; from the second point toward the first point is the *opposite* direction.

6. An **angle** is a *difference* of direction.

7. A **line** is that which has length, but not breadth, or thickness.

It has also position and directions.

8. A **straight line** is a line that does not change its directions at any point; it has but two directions, each the opposite of the other.

9. A **broken line** is a line made up of several straight lines, no one of which has the direction of any adjoining line.

All points lying in the same two direc... are to be called ...

10. A **surface** is that which has ... and breadth, but ... thickness.

The edges of a box are lines; ...
Point out some surfaces ...

11. A **plane** is a ... are joined by a straight ... surface. A plane ... straight edges ... or irregular.

12. The vertex of a plane

lines meet.

13. A plane figure is a

lines.

The

to be plane figures.

ANGLES.

15. A solid is that

ness. The space,

space, is intended.

16. Geometry

surfaces, figures

17. Solids

lines coincide in direction.

straight lines coincide in

qual angles coincide.

equal angles and respectively

two angles and the included side of

respectively equal angles and the

figure.

TION II.—ANGLES.

OREM: If one straight line meets another straight

point not an extremity, the sum of the adjacent angle

to two right angles.

I. THEOREM: If two straight lines intersect, the vertical

gles are equal.

f the theorems and problems is given at the beginning

books, not only for purposes of review but so that

possible, be induced to make demonstrations of their

assistance of the demonstrations in the text.

24. The **demonstration** or **proof** is the formal course of reasoning used to establish the truth of a proposition.

25. The **conclusion** is the final announcement of a theorem as proved, or of a problem as solved.

26. A **corollary** is an obvious consequence of a proposition.

27. A **scholium** is a remark on one or more propositions.

AXIOMS.*

28. An **axiom** is a self-evident truth ; as,

Axiom I. Things equal to the same thing are equal to each other.

It follows, that : *a*. Things equal to equal things are equal to each other. *b*. Things that may coincide in all their parts are equal to each other. *c*. Things that are equal may be substituted, in expressions of extent, one for the other.

Axiom II. The whole is equal to the sum of all its parts. The whole is, therefore, greater than any of its parts.

Axiom III. Equals added to equals give equals.

Axiom IV. Equals subtracted from equals give equals.

Axiom V. Equals multiplied by equals give equals.

Axiom VI. Equals divided by equals give equals.

Axiom VII. Equal powers of equals are equals.

Axiom VIII. Equal roots of equals are equals.

POSTULATES.*

29. A **postulate** is a self-evident problem ; as,

Post. I. One straight line, and only one, can be drawn between any two points.

Post. II. A straight line is the shortest line that can be drawn between any two points.

* For additional axioms, see p. 75 ; for additional postulate, p. 23.

BOOK I.—STRAIGHT LINES AND ANGLES.

SYNOPSIS.*

SECTION I.—SUPERPOSITION.

30. 1. PROPOSITIONS.

PROP. I. PROBLEM: To make two straight lines coincide in direction.

PROP. II. PROBLEM: To make two equal straight lines coincide in position.

PROP. III. PROBLEM: To make two equal angles coincide.

PROP. IV. PROBLEM: To make two equal angles and respectively equal sides coincide.

PROP. V. PROBLEM: To make two angles and the included side of one figure coincide with respectively equal angles and the included side of another figure.

2. MISCELLANEOUS EXERCISES.

SECTION II.—ANGLES.

31. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. VI. THEOREM: If one straight line meets another straight line at a point not an extremity, the sum of the adjacent angles is equal to two right angles.

PROP. VII. THEOREM: If two straight lines intersect, the vertical angles are equal.

---

* A statement of the theorems and problems is given at the beginning of the respective books, not only for purposes of review, but so that students may, if possible, be induced to make demonstrations of their own without the assistance of the demonstrations in the text.

PROP. VIII. THEOREM : If two straight lines meet a third straight line at the same point, making the sum of the contiguous angles one each side of the third straight line equal to two right angles, the two lines that thus meet form one and the same straight line.

PROP. IX. If two straight lines, on opposite sides of a third straight line, meet at a point in that line making the vertical angles equal, the two straight lines that thus meet form one and the same straight line.

PROP. X. If two straight lines intersect, making one of the angles at the point of intersection a right angle, the three other angles will also be right angles.

SECTION III.—PERPENDICULAR LINES.

32. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XI. THEOREM : From a given point on a straight line, only one perpendicular to that line can be erected.

PROP. XII. THEOREM : From a given point without a straight line, only one perpendicular to that line can be drawn.

PROP. XIII. THEOREM : If two straight lines are perpendicular to the same third straight line, they are parallel.

PROP. XIV. THEOREM : A straight line perpendicular to one of two parallel straight lines is also perpendicular to the other.

PROP. XV. THEOREM : If a straight line is perpendicular to another straight line at its middle point, it is everywhere equally distant from the extremities of that line.

3. MISCELLANEOUS EXERCISES.

SECTION IV.—PARALLEL LINES.

33. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XVI. THEOREM : If two parallel straight lines intersect a third straight line, the alternate angles are equal.

PROP. XVII. THEOREM : If two parallel straight lines intersect a third straight line, the opposite exterior and interior angles on the same side are equal, and the two interior angles on the same side are together equal to two right angles.

PROP. XVIII. THEOREM: If two straight lines intersect a third straight line, making the alternate angles equal, the two straight lines are parallel.

PROP. XIX. THEOREM: If two straight lines are parallel to the same third straight line, they are parallel to each other.

PROP. XX. THEOREM: Angles made by straight lines extending in respectively the same parallel directions are equal.

SECTION V.—OBLIQUE LINES.

34. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XXI. THEOREM: A perpendicular is the shortest line that can be drawn from a given point to a given straight line.

PROP. XXII. THEOREM: If a perpendicular and oblique lines are drawn from a given point to the same straight line, any two oblique lines, meeting the given line at points equally distant from the foot of the perpendicular, are equal.

PROP. XXIII. THEOREM: If from a given point without a straight line a perpendicular and several oblique lines are drawn to that line, of two oblique lines that meet the given line at different distances from the foot of the perpendicular, the one that meets the given line at the greater distance is the longer line.

PROP. XXIV. THEOREM: If two oblique straight lines drawn from a given point to a given straight line are equal, they meet the given line at points equally distant from the foot of the perpendicular drawn from the same point to the same straight line.

PROP. XXV. THEOREM: If two oblique straight lines from the same point to the same straight line are unequal, they meet the given line at points unequally distant from the foot of the perpendicular from the same point to the same straight line; and the longer line meets the given line at the greater distance from the foot of that perpendicular.

3. MISCELLANEOUS EXERCISES.

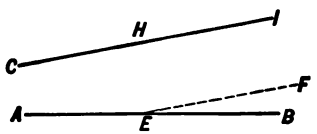
GENERAL SCHOLIA.

35. 1. THE THEORY OF LOCI. 2. THE SCIENCE OF GEOMETRY.

SECTION I.—SUPERPOSITION.

PROPOSITION I.

36. Problem: ***To make two straight lines coincide in direction.***



Statement: Let *A B* and *C I* be any two straight lines. It is required to make them coincide in direction.

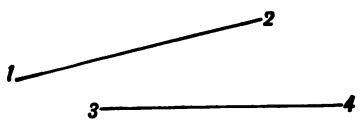
Construction: Place any point of the line *C I*, as the point *C*, upon any point in the line *A B*, as the point *A*, and then move *C I* until some other point, as *H*, falls on *A B*, as at *E*; that is, cause the lines to have two points in common.

Demonstration: *At A* and *E*, the lines coincide, by construction. *Between A* and *E*, the two lines coincide; for between two points only one straight line can be drawn. (POST I.) *Beyond A* and *E*, the lines coincide in direction; for, if they do not, they must at some point separate, taking different directions, as *E F* and *E B*. If they separate, one line or the other must change its directions. Both lines, however, are straight lines, and no straight line can change its directions. (DEF.) The lines then coincide in direction beyond the points *A* and *E*. They must, therefore, coincide throughout in direction.

Conclusion: *A B* and *C I* being any two straight lines, it follows, in general, that: Any two straight lines may be made to coincide in direction by being made to have two points in common.

PROPOSITION II.

37. Problem: ***To make two equal straight lines coincide in position.***



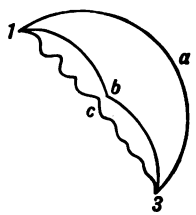
Statement: Let the lines *1 2* and *3 4* be any two equal straight lines. It is required to make them coincide in position.

Construction: Place the line *1 2* upon the line *3 4*, so that the point *1* shall fall upon the point *3*, and so that the line *1 2* shall take the direction of the line *3 4* (PROP. I.) ; that is, so that two terminal points shall coincide, and the lines shall have the same direction.

Demonstration: The point *1* having fallen upon the point *3*, and the line *1 2* upon the line *3 4*, the point *2* will fall somewhere in the line *3 4*. (PROP. I.) If the line *1 2* were shorter than the line *3 4*, the point *2* would fall short of the point *4*. If the line *1 2* were longer than the line *3 4*, the point *2* would fall beyond the point *4*. As the line *1 2* is neither shorter nor longer than the line *3 4*, the point *2* falls neither to the one side nor to the other of the point *4*, but upon that point.

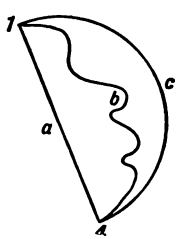
Conclusion: *1 2* and *3 4* being any two equal straight lines, it follows, in general, that: Any two equal straight lines may be made to coincide in position by placing two extremities together, and making the lines coincide in direction.

2. Exercise II. *With lines:* 1. Between any two points draw several lines, as between the points *1* and *3*.



1. What has each line besides position and direction? 2. Are the lines of equal length? 3. What is, then, the peculiar property of lines? Length. 4. The marks have breadth, but the lines are supposed to have no breadth.

2. Indicate any two points, as the points *1* and *4*, and, placing the ruler so that the two points are against the same edge, draw a line from the one point to the other: viz., the line *1 a 4*; draw also other lines between *1* and *4*, as *1 b 4*, and *1 c 4*.



1. What kinds of lines are the different lines thus drawn? 2. Which of the lines is made without change of direction? 3. What kind of a line is it? 4. Which line is a straight line? 5. Which line might be called a crooked line, and which a curved line? 6. Which line seems to be made with irregular changes of direction? 7. Which line seems to be made with regular changes of direction?

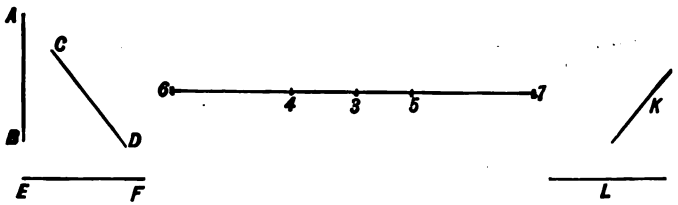
3. Draw a straight line between any two points.

1. Draw, if possible, a shorter line between the same two points. 2. Which line, then, is the shortest line that can be drawn between any two points?

4. From one point to another, or from one point in the direction of another point, draw a straight line.

1. Draw, if possible, another straight line, either (*a*) between the same two points, or (*b*) from the one point in the direction of the other point. 2. How many straight lines can be drawn between the same two points? 3. How many straight lines can be drawn from one point in the direction of another point? 4. What will determine, then, the directions of a straight line?

5. By using the rule, and also by using the compass or dividers, compare the lengths of any two straight lines.

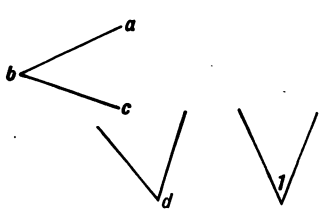


1. Is the line *K* or the line *L*, the longer, by the compass and by the rule? 2. How long is the line *K*? How long is the line *L*? 3. Show by the dividers that two of the lines on this page are equal. 4. Show by the rule that two other lines are equal.

6. By the dividers, lay off one line on another, or take a part of one line equal to another line.

1. From *6* lay off *A B* on the line *6 7*. What part of the line does it cover? 2. Lay off the same line on *6 7* from *7*. 3. In *4 7* take any point *3*, and cut off a part of *3 7* equal to *3 4*.

3. Exercise III. *With angles:* 1. Draw a straight line from any point in any direction, and from the same point draw a straight line in any other direction.



1. What name shall be given to the difference of direction between two straight lines that meet? 2. What is an angle? 3. Where is the angle *a b c*? *c b a*? *d*? *1*? 4. Make a number of angles and indicate their names in different ways. 5. Point to different angles in the structure of the recitation room, furniture, etc.

2. From any given point, draw a number of straight lines.

1. Indicate all the angles thus made at the given point. 2. How many angles can be made at a given point?

DEFINITIONS.

4. A **point** is that which has position only.

5. **Direction** is relative position of points.

From one point toward a second point is one direction; from the second point toward the first point is the *opposite* direction.

6. An **angle** is a *difference* of direction.

---

7. A **line** is that which has length, but not breadth, or thickness.

It has also position and directions.

8. A **straight line** is a line that does not change its directions at any point; it has but two directions, each the opposite of the other.

9. A **broken line** is a line made up of several straight lines, no one of which has the directions of any adjoining line.

All points lying in the same two directions are said to be *collinear*.

---

10. A **surface** is that which has length and breadth, but not thickness.

The *edges* of a box are lines; and the sides of the box are *surfaces*. Point out some surfaces in the school-room.

11. A **plane** is a surface such that, if any two of its points are joined by a straight line, that line will lie wholly in the surface. A *plane surface* is a flat surface. By the use of a straight edge, a surface may be found to be *plane*, or *curved*, or *irregular*, or it may be found to be a *rough* surface.

12. A **plane angle** is an angle in a plane. It is the difference in direction between two straight lines that meet, or that may meet, in a plane.

13. The **vertex** of a plane angle is the point at which the lines meet.

14. A **plane figure** is a portion of a plane bounded by lines.

The figures of the first seven books in this volume are all assumed to be plane figures.

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15. A **solid** is that which has length, breadth, and thickness. The space, rather than the substance occupying the space, is intended.

16. **Geometry** is the science of space; it treats of solids, surfaces, figures, lines, and angles.

17. Solids, surfaces, figures, lines, and angles are called **geometrical magnitudes.**

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18. A **proposition** is that which is offered for consideration.

The parts of a proposition are the following: 1. *Theorem* or *Problem;* 2. *Statement;* 3. *Construction;* 4. *Demonstration;* 5. *Conclusion;* 6. *Corollaries;* 7. *Scholium.* In addition, there may be a *discussion* of the limiting conditions. In general, however, a proposition is divided into the first five parts.

19. A **theorem** is something to be proved.

Every theorem consists of two parts: (1) *Hypothesis,* or condition; and (2) *Conclusion.*

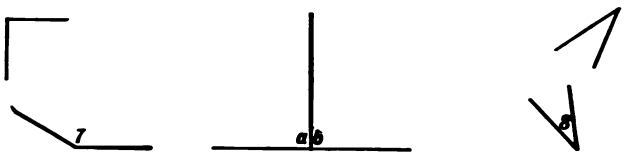
20. If the conclusion of one theorem is used as the condition of another, the condition of the first being the conclusion of the second, each theorem is the **converse** of the other.

21. A **problem** is a question proposed for solution.

22. The **statement** announces the theorem or problem with respect to some special figure.

23. The **construction** is the drawing of additional lines or figures used in the proposition, or the placing of figures in any desired position.

SECTION II.—ANGLES.



DEFINITIONS.

42. Adjacent angles are angles formed by one straight line meeting another straight line at a point not an extremity. If the two adjacent angles are equal, each is a **right angle**; and each line is **perpendicular** to the other line.

43. An angle that is not a right angle is an **oblique angle.**

44. An **obtuse angle** is an angle greater than a right angle, as the angle *7*.

45. An **acute angle** is an angle less than a right angle, as the angle *8*.

46. A **complementary angle,** or the **complement** of an angle, is the angle, which, added to a given angle, makes the sum equal to one right angle.

47. A **supplementary angle,** or the **supplement** of an angle, is the angle, which, added to a given angle, makes the sum equal to two right angles.

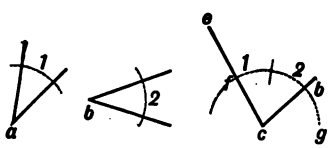
RULE AND COMPASS.

48. *To make an angle equal to a given angle.*

Placing one point of the dividers on *a*, the vertex of the given angle, extend the pencil point of the dividers to any distance, and describe an arc (or curve), as *b c*, between the two sides of the given angle. With the stationary point of the dividers on *f*, the given point on any given straight line, and the same distance between the points, draw an arc *e g*, of indefinite length, from the given line.

Measure with dividers from *b* to *c*, and cut off an equal portion, *e h*, of the arc *e g*. Draw *f h*; it makes the angle *f* equal to the angle *a*.

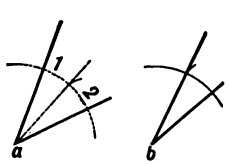
49. *To construct an angle equal to the sum of two angles.*



Let *a* and *b* be the given angles, and let *c e* be any straight line. It is required at the point *c* to construct an angle equal to the sum of the angles *a* and *b*.

With any distance between the points of the dividers, draw the arcs *1* and *2*, and with the same distance an arc *f g*, of indefinite length.

On the arc *f g* measure with the dividers a part, *1*, equal to *1*, the arc of the angle *a*, and beyond this, *2*, equal to *2*. Draw the line *c b*. The angle *f c b* is equal to the sum of the angles *a* and *b*.

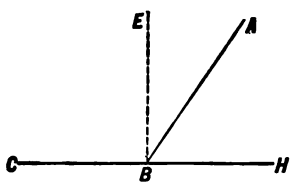


50. *To construct an angle equal to the difference of two angles.*

In the greater angle, *a*, construct an angle, *2*, equal to the smaller angle, *b*. The angle *1* is the required difference.

PROPOSITION VI.

51. Theorem: ***If one straight line meets another straight line at a point not an extremity, the sum of the adjacent angles is equal to two right angles.***



Statement: If the straight line *A B* meets the straight line *C H* at *B*, any point of *C H* not an extremity, the sum of the angles *A B C* and *A B H* is equal to two right angles. If *A B* were perpendicular to *C H*, demonstration would be unnecessary. The angles would be right angles, by definition.

Construction: If *A B* is not perpendicular to *C H*, draw *B E* perpendicular to *C H* at the point *B*.

Demonstration: By construction,

$$C\,B\,E + E\,B\,H = 2 \text{ right angles;}$$

but,
$$E\,B\,H = E\,B\,A + A\,B\,H; \quad \text{(Axiom II.)}$$

therefore,
$$C\,B\,E + E\,B\,A + A\,B\,H = 2 \text{ right angles;} \quad \text{(Axiom I.)}$$

but,
$$C\,B\,E + E\,B\,A = A\,B\,C; \quad \text{(Axiom II.)}$$

therefore,
$$A\,B\,C + A\,B\,H = 2 \text{ right angles.} \quad \text{(Axiom I.)}$$

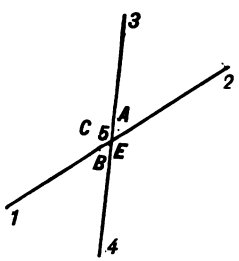
Conclusion: *A B* being any straight line, meeting *C H*, any other straight line, at a point not its extremity, it follows, in general, that: If one straight line meets, etc.

Corollary I. The sum of all the angles that can be made on the same side of a straight line, at any point in the line, without repetition, is equal to two right angles.

Corollary II. The sum of all the angles that can be made about a given point, without repetition, equals four right angles.

PROPOSITION VII.

52. Theorem: ***If two straight lines intersect, the vertical angles are equal.***



Statement: Let the straight lines *1 2* and *3 4* intersect at the point *5*. The angle *A* is to be proved equal to its opposite or vertical angle *B*, and the angle *C* to its opposite or vertical angle *E*.

Demonstration: The angle *A* plus the angle *C* equals two right angles; also, the angle *C* plus the angle *B* equals two right angles. (PROP. VI.) Therefore,

$$A + C = C + B. \qquad \text{(AXIOM I.)}$$

Subtracting the angle *C* from both members of the equation,

$$A = B. \qquad \text{(AXIOM IV.)}$$

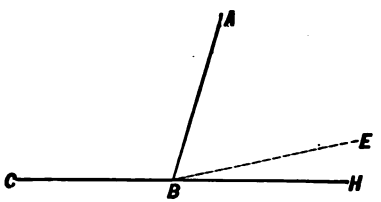
In similar manner, the angles *C* and *E* may be proved equal.

Conclusion: *1 2* and *3 4* being any two intersecting lines, etc.

Angles with a common vertex are *contiguous*, if they lie on opposite sides of one straight line; and *vertical*, if they lie on opposite sides of two straight lines. *Adjacent angles* are supplementary contiguous angles.

Proposition VIII.

53. Theorem: ***If two straight lines meet a third straight line at the same point, making the sum of the contiguous angles one each side of the third straight line equal to two right angles, the two lines that thus meet form one and the same straight line.***



Statement: Let the two straight lines *C B* and *H B* meet the third straight line *A B* at the point *B*, making *A B C* plus *A B H* equal to two right angles. It is to be proved that *C B* and *H B* are one and the same straight line.

Construction: If *B H* is not the continuation of *C B*, extend *C B* to *E*, making *C B E* a straight line.

Demonstration: If *C B E* be a straight line,

$$ABC + ABE = 2 \text{ right angles.} \quad \text{(Prop. VI.)}$$

By hypothesis, $ABC + ABH = 2$ right angles.

Then, $ABC + ABE = ABC + ABH$; (Axiom I.)

but, $ABC = ABC$; therefore,

by equal subtractions, $ABE = ABH$. (Axiom IV.)

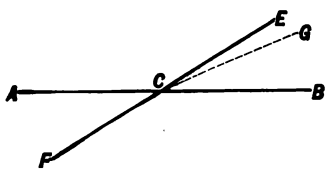
This, however, is absurd, as *A B E* is only a part of *A B H*, and a part cannot be equal to the whole. (Axiom II.)

The supposition that leads to an absurd conclusion must be abandoned. *B H* is, then, the continuation of *C B*, and they are one and the same straight line.

Conclusion: *C B* and *B H* being any two straight lines meeting *A B*, any third straight line, at the same point, etc.

PROPOSITION IX.

54. Theorem: ***If two straight lines on opposite sides of a third straight line meet at a point in that line, making the vertical angles equal, the two straight lines that thus meet form one and the same straight line.***



Statement: *E C* and *F C* meet *A B* on opposite sides at *C*, making *E C B* equal to *A C F*. Prove that *C E* and *C F* form one and the same straight line.

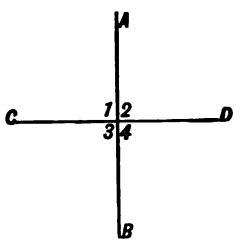
Construction: If *C E* is not the continuation of *C F*, extend *F C* to *G*.

Demonstration: If *F C G* be a straight line, *G C B* is equal to *A C F*, because they are opposite, or vertical angles. (PROP. VII.) *E C B*, which is, by hypothesis, equal to *A C F*, must then be equal to *G C B*, as things equal to the same thing are equal to each other. (AXIOM I.) *E C B* and *G C B* cannot, however, be equal, as the whole is greater than any of its parts. (AXIOM II.) The supposition that *C E* is not the extension of *F C* should, then, be abandoned. *C E* and *F C* must form one and the same straight line.

Conclusion: *C E* and *F C* being any two straight lines on opposite sides of *A B*, any third straight line, meeting at any point *C*, in *A B*, so as to make the vertical angles equal, it follows, in general, that: If, etc.

Proposition X.

55. Theorem: ***If two straight lines intersect, making one of the angles at the point of intersection a right angle, the three other angles will also be right angles.***



Statement: *A B* and *C D* intersect, making one of their angles, as the angle *1*, a right angle. Prove that the angles *2*, *3*, and *4* are also right angles.

Demonstration: The angles *1* and *3*, being adjacent angles, are together equal to two right angles. (Prop. vi.) The angle *1* being a right angle, by hypothesis, the angle *3* is, then, also, a right angle. (Axiom iv.) The angles *2* and *4* are opposite, or vertical, respectively, to the angles *3* and *1*; and they are, therefore, respectively equal to them. (Prop. vii.) As *1* and *3* are right angles, the angles *2* and *4* must, also, be right angles. (Axiom i.)

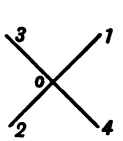
Conclusion: *A B* and *C D* being any two straight lines, etc.

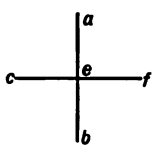
Scholium. The angular space about any point in a plane is conceived to be divided into 360 equal parts, or degrees (°); each degree, into 60 equal minutes (′); each minute, into 60 equal seconds (″). Each right angle, then, contains 90°.

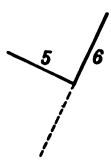
SECTION III.—PERPENDICULAR LINES.

DEFINITIONS.

56. Perpendicular lines are lines that meet or intersect so as to make the adjacent angles equal; as *1 2* and *3 4*.





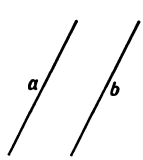


Either line is said to be perpendicular to the other. The adjacent angle is sometimes made by producing one of the sides.

57. A **right angle** is an angle made by two lines that are perpendicular to each other; as any of the angles at *e* or *o*.

Point to right angles in the structure of the room, the furniture, books, etc., and indicate the perpendicular lines.

58. Lines that lie in the same plane, and that cannot meet, even if produced, are **parallel lines**; thus, *a* and *b* are *parallel* lines.



With any point in *a* fixed, any change, within the plane, in the directions of *a* will cause one end to incline *toward* and the other end *from b*.

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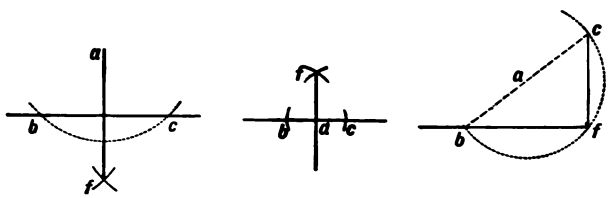
59. Postulate III. Through a given point only one straight line can be drawn parallel to a given line.

For other postulates, see p. 6.

RULE AND COMPASS.

60. *To draw one line perpendicular to another :* (1) From any point without the given line ; (2) from any point on the line ; (3) from the extremity of the line.



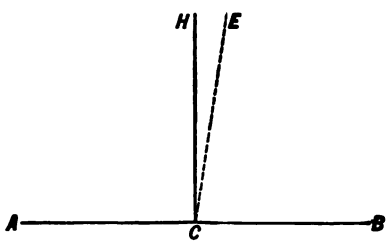
1. With one leg of the dividers at the given point, *a*, draw a curved line with the pencil at the other point of the dividers, so as to cut the given line in two points, *b* and *c*. With the same, or any other distance between the points of the dividers, and with the points *b* and *c* as centers (stationary points), draw two curves that shall intersect, as at *f*. Place either edge of the ruler against *a* and *f*, and draw *a f*. The straight line *a f* is perpendicular to the straight line *b c*.

2. Place one leg of the dividers on *a*, and with the other leg measure off on the given line any two equal distances, as *a b* and *a c*. With *b* and *c* as centers, and any distance greater than *a b* between the points of the dividers, draw two arcs intersecting, as at *f*. With the ruler drawing *a f*, it is the perpendicular required.

3. Take any point, *a*, without the given line. Open the dividers to the distance *a f*. Draw a curve intersecting the given line at *b*. Draw *b a*, and produce it to cut the curve at *c*. Draw *c f ;* it is perpendicular to the given line at *f*, its extremity.

PROPOSITION XI.

61. Theorem: ***From a given point on a straight line only one perpendicular to that line can be erected.***



Statement: Let *A B* be any straight line, one of whose points is *C*. From *C* let *C H* be drawn perpendicular to *A B*.

Construction: If possible, from the same point *C*, and on the same side of *A B*, let *C E* be erected perpendicular to *A B*.

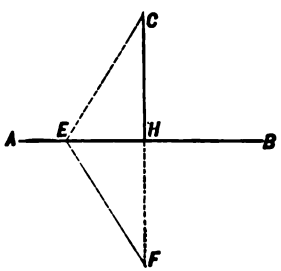
Demonstration: If *E C* be perpendicular to *A B*, the angle *E C B* is a right angle; and, as *C H* is perpendicular to *A B*, the angle *H C B* is a right angle. (DEF.) The angle *E C B* is, therefore, equal to the angle *H C B*. (AXIOM I.) This is absurd, for a part cannot be equal to the whole. (AXIOM II.) As the supposition that leads to an absurdity must be abandoned, it is not true that from the same point *C* more than one perpendicular to *A B* can be erected.

Conclusion: *C* being any given point on *A B*, any straight line, it follows, in general, that: From a given point on a straight line only one perpendicular to that line can be drawn.

PROPOSITION XII.

62. Theorem: ***From a given point without a straight line, only one perpendicular to that line can be drawn.***

Statement: Let *A B* be a straight line, and let *C* be a point without it. Only one straight line, as *C H*, can be drawn from *C* perpendicular to *A B*.



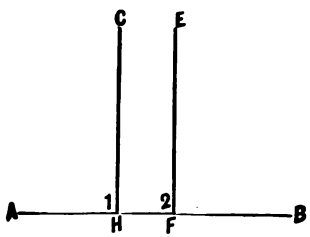
Construction: If possible, let *C E*, a second perpendicular to *A B*, be drawn. Extend *C H*, making *H F* equal to *C H*, and draw *E F*.

Demonstration: The angles *C H E* and *E H F* are equal, each being a right angle. (PROP. X.) The sides about these equal angles are equal, by construction. The figure *E H F* may then be made to coincide with *E H C*. (PROP. IV.) *F* having fallen on *C*, *E F* will coincide with *E C*. (POST. I.) The angles at *E* are thus proved equal; but *C E H* is, by supposition, a right angle; therefore, *F E H* is a right angle. (AXIOM I.) *C E*, *E F* is, then, a straight line, as the adjacent angles at *E* are together equal to two right angles. (PROP. VIII.) This is absurd, for then there would be two straight lines between the same two points, *C* and *F*. (POST. I.) The supposition that two perpendiculars can be drawn must be abandoned.

Conclusion: *C* being any point without *A B*, which is any straight line, it follows, in general, that: From, etc.

PROPOSITION XIII.

63. Theorem: ***If two straight lines are perpendicular to the same third straight line, they are parallel.***



Statement: Let the straight lines *C H* and *E F* be each perpendicular to the straight line *A B*. *C H* and *E F* are to be proved parallel to each other.

Demonstration: If *C H* and *E F* are not parallel, they will meet on one side or the other of *A B*. If they meet, there will then be two perpendiculars drawn from the same point without a straight line to that line. This, however, is impossible. (PROP. XII.) The supposition that *C H* and *E F* are not parallel must be abandoned. The lines *C H* and *E F* are, then, parallel to each other.

Conclusion: *C H* and *E F* being any two straight lines, etc.

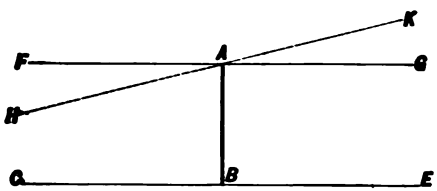
Exercises. 1. The lines that bisect two adjacent angles are perpendicular to each other.

2. What is the complement of $25°$? Of $37° 45' 59''$?

3. Find the supplement of $125°$. Of $37° 45' 59''$.

Proposition XIV.

64. Theorem: ***A straight line perpendicular to one of two parallel straight lines is also perpendicular to the other.***



Statement: *F G* and *C E* are parallel, and *A B* is perpendicular to *C E*, one of the two lines. Prove that *A B* is also perpendicular to *F G*.

Construction: If *A B* is not perpendicular to *F G*. draw *H K* so that *A B* shall be perpendicular to *H K* at *A*.

Demonstration: If *H K* be perpendicular to *A B*, *H K* and *C E* are parallel, both being perpendicular to the line *B A*. (Prop. xiii.) *F G* is, however, by hypothesis, parallel to *C E*. There are, then, two straight lines, *F G* and *H K*, through the same point *A*, parallel to the same straight line *C E*; but this is impossible. (Post. iii.) The supposition that *A B* is not perpendicular to *F G* must, then, be abandoned; and it follows that *A B* is perpendicular to *F G*.

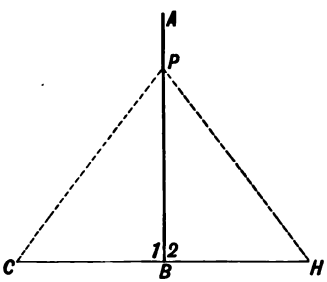
Conclusion: *A B* being any straight line perpendicular, etc.

Corollary. Two straight lines, perpendicular each to one of two parallel straight lines, are themselves parallel.

PROPOSITION XV.

65. Theorem: ***If a straight line is perpendicular to another straight line at its middle point, it is everywhere equally distant from the extremities of that line.***

Statement: Let the straight line *A B* be perpendicular to the straight line *C H*, at its middle point *B*. Every point in *A B*, as *P*, is equally distant from *C* and *H*, the extremities of *C H*.



Construction: Draw the lines *P C* and *P H*, representing the distances of *P* from the points *C* and *H*.

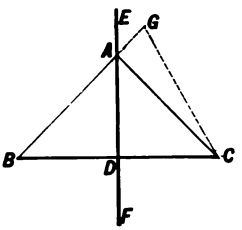
Demonstration: Fold the figure *P B H* on *P B* as an axis, so that it shall fall upon the figure *P B C*. Since the angle *2* is equal to the angle *1*, each being a right angle, the line *B H* coincides in direction with the line *C B* (PROP. III.); and, since *B H* equals *B C*, the point *H* will fall upon the point *C*. (PROP. II.) As the point *H* falls upon the point *C*, and the point *P* remains common, the straight lines *P C* and *P H*, the lines representing the distances of *P* from *C* and *H*, coincide in position, and they are, therefore, equal. (POST. I.)

Since *P* is any point on the line *A B*, and it is equally distant from the extremities of *C H*, every point on the line *A B* is equally distant from the extremities of *C H*.

Conclusion: *A B* being any straight line perpendicular to *C H*, any other straight line, at its middle point, it follows, in general, that: If a straight line, etc.

Corollary I. Every point without the bisecting perpendicular is unequally distant from the extremities of the given line.

If G is any point without the perpendicular, its distance from C, measured on the straight line GC, is less than the distance measured on the broken line GA, AC (POST. II.); or, substituting AB for its equal, AC, GC is less than GA, AB; that is, GC is less than GB.



Corollary II. If two points are each equally distant from the extremities of a given line, any line joining the two points will bisect the given line at right angles.

Each of the two points must be in the bisecting perpendicular, and the two points together determine the perpendicular. Construct figures and illustrate.

Corollary III. If a point is equally distant from the extremities of a given line, any straight line joining it and the middle point of the given line is perpendicular to that line.

MISCELLANEOUS EXERCISES.

66. 1. A straight line that bisects (divides equally) one of two vertical angles bisects also the other angle.

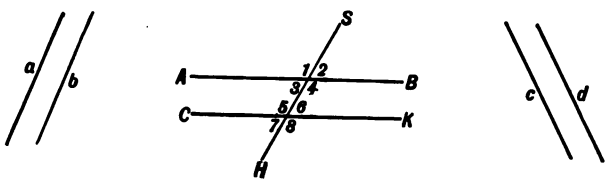
2. A line drawn through the vertex of an angle, perpendicular to the bisectrix of the angle, makes equal angles with the sides of the given angle.

3. Two straight lines, bisecting adjacent angles, are perpendicular to each other.

4. Bisect two adjacent angles and indicate all the angles that are *complementary*, and all that are *supplementary*.

5. If complementary contiguous angles are bisected, what is the value of the angle made by the bisectrices?

SECTION IV.—PARALLEL LINES.



DEFINITIONS.

67. Parallel lines are lines that lie in the same plane, and that cannot meet, even if produced.

68. A straight line that intersects two or more straight lines is said to be a **secant**; it is also said to be a **transversal**; thus, ***S H*** is a *secant*, or *transversal.*

A B and ***C K*** are parallel straight lines cut by the transversal ***S H.***

69. Adjacent angles are angles formed by one straight line meeting another straight line at a point not an extremity.

The angles *1* and *2* are *adjacent;* also, *1* and *3; 3* and *4; 4* and *2,* etc. All adjacent angles are contiguous, but all contiguous angles are not adjacent.

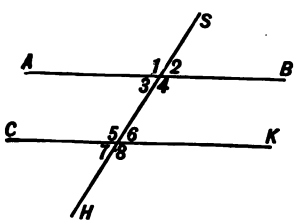
70. Opposite angles are any two angles, not adjacent, formed by one straight line intersecting one or more other straight lines.

The angles *2* and *6, 3* and *7, 4* and *8, 1* and *5* are opposite angles. The angles *1* and *4* are *opposite or vertical angles;* also, *2* and *3; 5* and *8; 6* and *7.*

71. Interior angles are the angles made by any two lines and that part of the transversal which is *between* the two lines.

The angles *3*, *4*, *5*, and *6* are *interior angles*.

72. Exterior angles are the angles made by any two lines and the parts of the transversal *without*, or *beyond*, the two lines.



The angles *1*, *2*, *7*, and *8* are *exterior angles*.

73. Angles on either side of the transversal are said to be **angles on the same side**, as the angles *1*, *3*, *5*, and *7*; angles on different sides of the transversal are said to be **angles on opposite sides**, as the angles *3* and *6*, etc.

74. In describing any two of the eight angles made by a secant cutting two parallel straight lines, note whether they are: (1) **Adjacent** or **opposite**; (2) **interior** or **exterior**; (3) on the **same side** or on **opposite sides** of the secant. Such angles are generally described with respect to these three points in succession.

1. The angles *1* and *3* are adjacent exterior and interior angles on the same side; *1* and *4* are opposite exterior and interior angles, on opposite sides; *1* and *5* are opposite exterior and interior angles, on the same side, etc. 2. What are *2* and *6*? *4* and *8*? *3* and *7*? *1* and *7*? *1* and *8*?

75. Alternate angles are *opposite interior* angles, on *opposite sides* of the transversal; as, *3* and *6*; and, also, *4* and *5*.

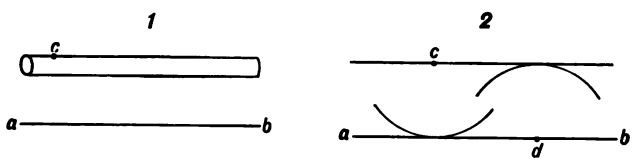
In the figure, the lines *A B* and *C K*, extending on either side of the transversal, lie in the *same parallel* directions; and, if one is supposed to extend on one side of the transversal, and the other on the other side, the lines lie, or extend, in *opposite parallel* directions. $2 = 50°$; $6 = 50°$; *1*, *3*, *4*, *5*, *7*, *8* = ?

RULE AND COMPASS.*

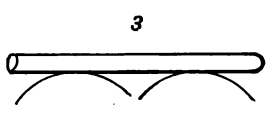
76. *To draw one line parallel to another:* (1) with a round ruler; (2) with a rule and a compass.

1. Place the round ruler in line with *a b*, the given line, and roll the ruler evenly to any distance, or against any given point, and then draw the required line against the edge of the ruler.

2. Place one leg of the dividers on *c*, the given point, and find with the other leg the nearest point in the given line;



lay off this distance from *d*, any other point in the given line, as in figure 2. Placing the ruler against *c* and the curved line last drawn, draw the required line; **or**

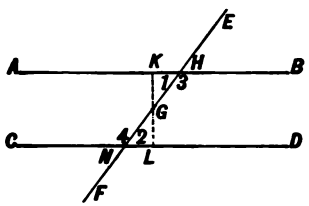


3. As in figure 3, from any two points in the given line draw curved lines with the pencil point of the compass at the equal distances from the given lines; and then draw the required line, by placing the ruler against the two curved lines thus drawn.

* The exercises under Rule and Compass are intended to enable students to draw correctly the figures required in following sections. They are intended to serve a practical purpose, and may or may not involve more knowledge than the student has acquired.

PROPOSITION XVI.

77. Theorem: ***If two parallel straight lines intersect a third straight line, the alternate angles are equal.***



Statement: Let the parallel straight lines *A B* and *C D* intersect the third straight line *E F*. The alternate angles *1* and *2*, or *3* and *4*, are to be proved equal to each other.

Construction: From *G*, the middle point of *N H*, the part of the transversal between the parallel lines, draw *G L* perpendicular to *C D*, and produce it to *K*.

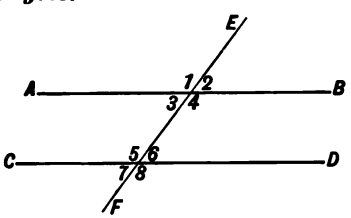
Demonstration: The angles at *K* are right angles. (PROP. XIV.) The angles at *G* are equal, being vertical angles. (PROP. VII.) Turn the figure *G N L* upon the figure *K G H*, so that *G* remains common, and *G L* shall take the direction of *G K*. Since the angles at *G* are equal, *G N* will take the direction of *G H*; and since *G N* equals *G H*, *N* will fall on *H*. *N L* and *H K*, being perpendicular to *G L* and *G K*, must, then, coincide; for, from a point without a straight line, only one perpendicular can be drawn to that line. (PROP. XII.)

As the figures coincide, the angles *1* and *2* are equal; and as *3* and *4* are the supplements of these angles, they, also, are equal. (AXIOM IV.)

Conclusion: *A B* and *C D* being any two parallel, etc.

PROPOSITION XVII.

78. Theorem: ***If two parallel straight lines intersect a third straight line, the opposite exterior and interior angles on the same side are equal, and the two interior angles on the same side are together equal to two right angles.***



Statement: *A B* and *C D* are parallel straight lines cut by the transversal *E F*. Prove that the opposite exterior and interior angles on the same side, as *2* and *6*, are equal, and that the two interior angles on the same side, as *4* and *6*, are together equal to two right angles.

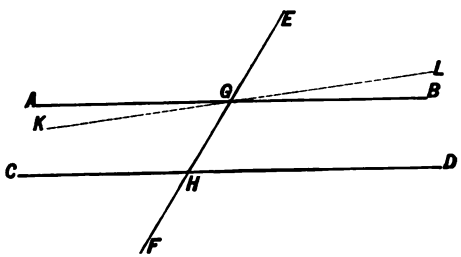
Demonstration: The alternate angles *3* and *6* are equal. (PROP. XVI.) The vertical angles *3* and *2* are equal. (PROP. VII.) The angles *6* and *2*, being each equal to *3*, are equal to each other (AXIOM I.); that is, the opposite exterior and interior angles on the same side are equal.

Since the angle *2* = the angle *6*, $2 + 4 = 6 + 4$. (AXIOM III.) The angles *2* and *4*, being adjacent angles, are together equal to two right angles. (PROP. VI.) The angles *6* and *4* must, then, together be equal to two right angles (AXIOM I.); that is, the two interior angles on the same side are together equal to two right angles.

Conclusion: *A B* and *C D* being any two parallel, etc.

PROPOSITION XVIII.

79. Theorem: ***If two straight lines intersect a third straight line, making the alternate angles equal, the two straight lines are parallel.***



Statement: *A B* and *C D* are straight lines, intersected by a third straight line, *E F*. The angles *A G H* and *G H D* are equal. Prove that *A B* and *C D* are parallel.

Construction: If *A B* is not parallel to *C D*, through *G* draw *K L* parallel to *C D*.

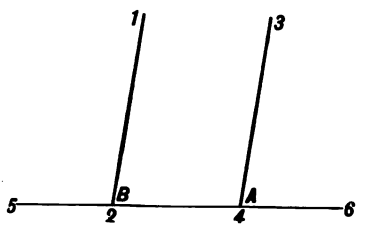
Demonstration: If *K L* be parallel to *C D*, the angle *K G H* is equal to the angle *G H D*, being alternate angles. (PROP. XVI.) By hypothesis, *A G H* is equal to *G H D*. *A G H* and *K G H*, each being equal to *G H D*, must be equal to each other. (AXIOM I.) This is absurd, as a part cannot equal the whole. (AXIOM II.) The supposition that *A B* is not parallel to *C D* is, then, false. *A B* is parallel to *C D*.

Conclusion: *A B* and *C H* being any two straight lines intersecting a third straight line, etc.

Corollary I. If the opposite exterior and interior angles on the same side are equal, the lines are parallel. (PROP. VII.)

Corollary II. If the two interior angles on the same side are together equal to two right angles, the lines are parallel.

Scholium I. If the exterior angle is greater than the opposite interior angle on the same side of a secant, the lines, if produced, will meet on that side.

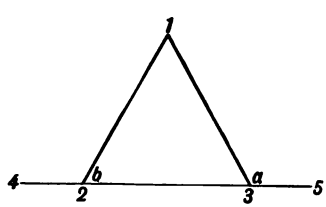


If the angle ***A*** is equal to the angle ***B***, the line ***4 3*** must be parallel to the line ***2 1***. If the angle ***A*** were greater than the angle ***B***, the line ***4 3*** would be inclined *toward* the line ***2 1***, and, if sufficiently extended, they would meet on that side of the line ***5 6***. This may, also, be seen from the fact, that, if ***A*** were greater than ***B***, the two interior angles would be less than two right angles.

Scholium II. If the exterior angle is less than the opposite interior angle on the same side, the lines will not meet on that side of the third straight line.

If the angle ***A*** were less than the angle ***B***, the line ***4 3*** would be inclined *from* the line ***2 1***, and, therefore, they could not meet. This may, also, be seen from the fact, that the two interior angles would then be together greater than two right angles.

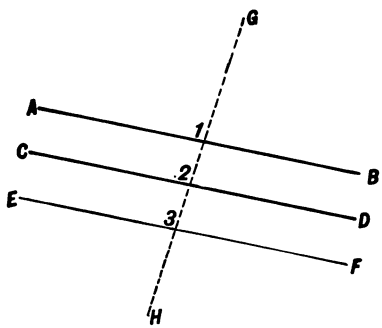
Scholium III. If two lines are drawn from the same point to the same straight line, the exterior angle is greater than the opposite interior angle on the same side.



If ***a*** were *equal* to ***b***, the line ***3 1*** would be parallel to the line ***2 1***. If ***a*** were *less* than ***b***, the line ***3 1*** would be inclined from the line ***2 1***. Neither result being true, ***a*** must be greater than ***b***.

PROPOSITION XIX.

80. Theorem: ***If two straight lines are parallel to the same third straight line, they are parallel to each other.***



Statement: Let the straight lines *A B* and *C D* be each parallel to *E F*. *A B* and *C D* are then parallel to each other.

Construction: Draw the line *G H*, intersecting the three given lines.

Demonstration: Since *A B* is parallel to *E F*, the angle *1* is equal to the angle *3*; and, since *C D* is parallel to *E F*, the angle *2* is equal to the angle *3*. (PROP. XVII.) The angle *1* is, therefore, equal to the angle *2*. (AXIOM I.)

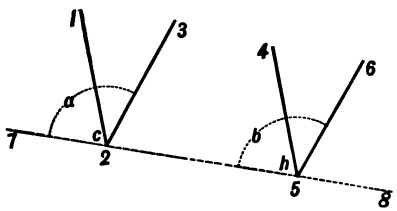
Since the angle *1* is equal to the angle *2*, the line *A B* is parallel to the line *C D*. (PROP. XVIII., COR. I.)

Conclusion: As *A B* and *C D* are any two straight lines, etc.

Corollary I. Straight lines passing through a given point parallel to a given line form one and the same straight line.

PROPOSITION XX.

81. Theorem: ***Angles made by straight lines extending in respectively the same parallel directions, are equal.***



Statement: Let the angles *1 2 3* and *4 5 6* be made by straight lines extending in the same parallel directions, *1 2* and *4 5* being parallel, and *2 3* and *5 6* being parallel. Prove that the angles *1 2 3* and *4 5 6* are equal.

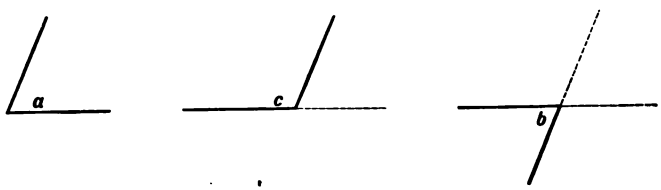
Construction: Through the vertices *2* and *5* draw the straight line *7 2 5 8.*

Demonstration: The angle *a* is equal to the angle *b*, as they are opposite exterior and interior angles on the same side of *7 8*; and the angle *c* is equal to the angle *h*, for the same reason. (PROP. XVII.) By equal subtractions, the given angle *2*, which is the difference between the angles *a* and *c*, is equal to the given angle *5*, which is the difference between the angles *b* and *h*. (AXIOM IV.)

Conclusion: As *1 2* and *4 5* are any parallel straight lines extending in respectively the same parallel directions, etc.

Scholium. If the angles are placed so that one partly includes the other, prove, as above, each equal to a third angle.

Corollary I. Angles made by straight lines extending in respectively opposite parallel directions are equal, as *a* and *b*.

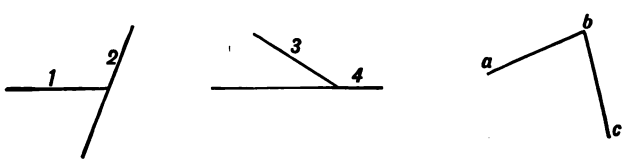


Each of the given angles is readily proved to be equal to the angle that is vertical to either one of them.

Corollary II. Angles made by straight lines extending, two in the same parallel directions and two in opposite parallel directions, are supplementary, as *a* and *c*, or *b* and *c*.

Either angle is readily proved to be equal to the angle adjacent to the other.

SECTION V.—OBLIQUE LINES.



DEFINITIONS.

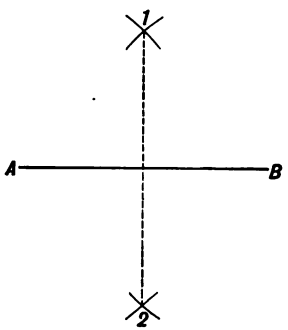
82. Oblique lines are lines that are not perpendicular or parallel to each other, as *1* and *2*; *3* and *4*; *a b* and *b c*.

83. Oblique lines include **oblique angles**, either *obtuse* or *acute*.

Point out the oblique angles among the figures, and indicate which are acute, and which, obtuse.

RULE AND COMPASS.

84. *To bisect any given straight line;* that is, to divide it into two equal parts.

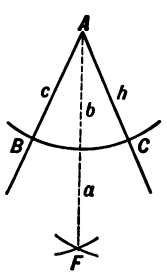


With any distance between the points of the compass greater than half of the given line, place one leg of the compass on one extremity of the line, and draw small arcs above and below the given line; and, in the same manner, draw corresponding arcs above and below the given line, from the other extremity as a center. With the ruler, draw the straight line *1 2*, joining the two points of intersection. The line *1 2* is perpendicular to *A B* at its middle point; it is, therefore, the line required.

85. *To draw a line perpendicular to a given line at its middle point.*

The same construction as in *1* gives the required line.

86. *To bisect any given angle.*

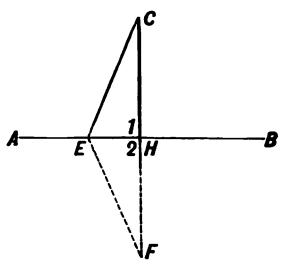


With the vertex of the given angle as a center, draw any arc between the sides, as the arc *B C*. With *B* and *C* as centers, and with the same distance between the compass points, draw two arcs, intersecting, as at *F*. With the ruler, draw the straight line *A F*. *A F* bisects the angle *A*, and it is, therefore, the required line.

PROPOSITION XXI.

87. Theorem: ***A perpendicular is the shortest line that can be drawn from a given point to a given straight line.***

Statement: Let *C H* be a perpendicular, and *C E* be any oblique line, both drawn to *A B*, any straight line, from *C*, any point without that line. *C H*, the perpendicular, is shorter than *C E*.



Construction: Extend the line *C H* to *F*, making *H F* equal to *C H*. Join *E* and *F* by the straight line *E F*.

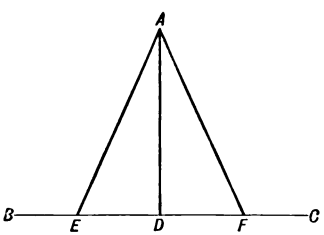
Demonstration: Since *A H* is perpendicular to *C F*, the angles *1* and *2* are right angles, and are, therefore, equal. (PROP. X.) Since the line *A H* is perpendicular to the line *C F* at its middle point *H*, the lines *E C* and *E F* are equal. (PROP. XV.) Since *C H F* is a straight line, it is the shortest distance between the two points *C* and *F*. (POST. II.) *C F* is, therefore, shorter than the broken line *C E*, *E F*; and *C H*, which is half of *C F*, is, therefore, shorter than *C E*, the half of *C E*, *E F*.

Conclusion: *A B* being any straight line, and *C* being any point without the line, it follows, in general, that: A perpendicular is the shortest line that can be drawn from a given point to a given straight line.

PROPOSITION XXII.

88. Theorem: ***If a perpendicular and oblique lines are drawn from a given point to the same straight line, any two oblique lines, meeting the given line at points equally distant from the foot of the perpendicular, are equal.***

Statement: Let *A* be a given point without the given straight line *B C*. Let *A D* be perpendicular to *B C*, and let *A E* and *A F* be two oblique straight lines meeting *B C* at *E* and *F*, any two points equally distant from *D*, the foot of the perpendicular *A D*. *A E* and *A F* are to be proved equal.



Demonstration: Since *E D* is equal to *D F*, the line *A D* is perpendicular to *E F* at its middle point *D*. *A* is, therefore, equally distant from *E* and *F*, and the oblique lines *A E* and *A F*, representing the distances, must be equal. (PROP. XV.)

Conclusion: *A* being any given point without *B C*, etc.

Exercises. 1. If four of five angles about a given point are, respectively, 20°, 30°, 40°, 50°, what is the fifth angle?

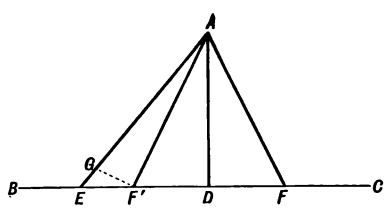
2. If three of four angles about a point are, respectively, 38° 42′, 51° 49″, and 75° 45′ 45″, what is the fourth angle?

3. If four of five angles about a point are, respectively, *a*, *b*, *c*, *d*, what is the fifth angle?

PROPOSITION XXIII.

89. Theorem: ***If from a given point without a straight line, a perpendicular and several oblique lines are drawn to that line, of two oblique lines that meet the given line at different distances from the foot of the perpendicular, the one that meets the given line at the greater distance is the longer line.***

Statement: Let *BC* be any straight line, and let *A* be any point without it. From *A* draw *AD* perpendicular to *BC*, and *AE* and *AF*, oblique lines from *A* to *BC*. Let *ED* be greater than *FD*. Prove that *AE* is longer than *AF*.



Construction: If *AF* is not on the same side of the perpendicular as *AE*, draw *AF'*, its equal, which shall be on the same side with *AE*. Draw *F'G* perpendicular to *AF'*.

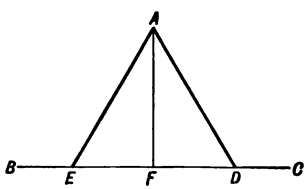
Demonstration: *AF'E* is greater than *ADF'*, or its equal, *AF'G*. (PROP. XVIII., SCHOL. III.) *F'G* falls, then, in the angle *AF'B*, and cuts the line *AE*, as at *G*. *AE* is greater than *AG* (AXIOM II.), which, being an oblique line, is greater than *AF'*, which is perpendicular to *F'G*. (PROP. XXI.)

As *AE* is greater than *AG*, and *AG* is greater than *AF'*, it is obviously true that *AE* is greater than *AF'*, or its equal *AF*.

Conclusion: *BC* being any straight line, and *AF* and *AE* any two oblique lines drawn from *A*, etc.

PROPOSITION XXIV.

90. Theorem: ***If two oblique straight lines drawn from a given point to a given straight line are equal, they meet the given line at points equally distant from the foot of the perpendicular drawn from the same point to the same straight line.***



Statement: *A F* is the perpendicular, and *A D* and *A E* are equal oblique lines. *F D* and *F E* are to be proved equal.

Demonstration: *F E* is not longer than *F D*, for then the line *A E* would be longer than the line *A D*, which is contrary to the hypothesis. (PROP. XXIII.) *F E* is not shorter than *F D*, for then the line *A E* would be shorter than the line *A D*, which is contrary to the hypothesis. (PROP. XXIII.) As *F E* is neither longer nor shorter than *F D*, it must be equal to *F D*.

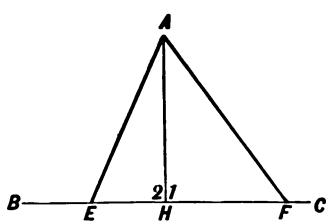
Conclusion: *A D* and *A E* being any two equal oblique, etc.

Corollary I. From a given point to a given line, only one line can be drawn equal to a given oblique line.

Corollary II. Equal oblique lines make equal interior and equal exterior angles with the transversal; and, also, with the perpendicular from the same point to the transversal. (Superposition, as in PROP. XV.)

Proposition XXV.

91. Theorem: ***If two oblique straight lines from the same point to the same straight line are unequal, they meet the given line at points unequally distant from the foot of the perpendicular from the same point to the same straight line; and the longer line meets the given line at the greater distance from the foot of the perpendicular.***



Statement: Let *B C* be any straight line, and *A*, any point without that line. Let *A H* be the perpendicular from *A* to *B C*, and *A F* and *A E*, be any two oblique lines, the line *A F* being longer than the line *A E*. *H F* is to be proved longer than *H E*.

Demonstration: If *H F* is not longer than *H E*, *H F* is equal to *H E*, or less than *H E*. *H F* cannot be equal to *H E*, for then *A F* would be equal to *A E*, which is contrary to the hypothesis. (Prop. xxii.) Nor can *H F* be less than *H E*, for then *A F* would be less than *A E*, which is contrary to the hypothesis. (Prop. xxiii.) As *H F* can be neither equal to nor less than *H E*, it is longer than *H E*.

Conclusion: As *A F* and *A E* are any two unequal oblique lines, etc.

Corollary I. Unequal oblique lines make unequal angles with the perpendicular to the transversal, and the longer line makes the greater angle.

Corollary II. The shorter oblique line makes the greater interior angle with the transversal.

Corollary III. The longer oblique line makes the greater exterior angle with the transversal.

Corollary IV. Oblique lines making equal angles with the perpendicular to the transversal are equal.

Corollary V. Oblique lines making equal angles with a transversal are equal.

Corollary VI. The oblique line making the greater angle with the perpendicular to the transversal is the longer line.

Corollary VII. The oblique line making the greater interior angle with the transversal is the shorter line.

Corollary VIII. The oblique line making the greater exterior angle with the transversal is the longer line.

MISCELLANEOUS EXERCISES.*

92. 1. Angles made by lines respectively perpendicular are equal or supplementary.

Through the vertex of one of the angles draw lines parallel to the sides of the other angle.

2. In a given straight line, find the point that is equidistant from two given points.

In what case is there no such point?

* It is important that the student indicate in writing the steps of his demonstrations for the theorems among the Miscellaneous Exercises.

3. If a straight line is perpendicular to the bisectrix of an angle at the vertex, it bisects the supplementary adjacent angle.

4. A line perpendicular at any point to the bisectrix of an angle makes equal angles with the sides of the given angle.

5 Through a given point draw a straight line parallel to a given line.

6. Through a given point draw a straight line parallel to a given line, by opposite exterior and interior angles on the same side.

7. Through a given point draw a straight line parallel to a given line, by two interior angles on the same side.

8. From a given point, draw a line making a required angle with a given line.

9. If two lines bisect vertical angles, they form one and the same straight line.

10. With a compass draw from a given point to a given straight line two lines of a given length, so that they shall be equally inclined to the given line.

11. Using a compass, draw from a given point to a given straight line two lines of a given length, so that their other extremities shall meet the given line at points equally distant from the foot of the perpendicular let fall from the given point to the given straight line.

12. Draw the shortest line possible from a given point to a given straight line, and prove that it is the shortest.

13. From a given point to a given straight line, with rule and compass, draw the line that shall have the greatest possible inclination to the given line, and prove that it has such inclination.

14. Draw a line that shall bisect a given line and be perpendicular to that line.

GENERAL SCHOLIA.

THEORY OF LOCI.

93. To determine the directions of a line, it has been seen that two points, or a point and a direction, must be known. To determine the position of a point, two lines must be known (PROP. V., COR. IV.); or, a direction and a given distance in the given direction will determine the position of the point.

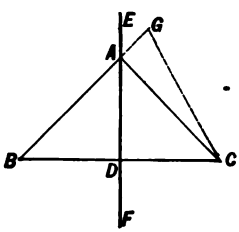
94. If the necessary premises, or conditions, are not known, the given problem will be *indeterminate*; thus, a line cannot be found, if the position of only one point is known; nor, can a point be found, if only one of the necessary lines is given.

95. Something may, however, be known, in indeterminate problems, even if the necessary conditions are not all given; thus, if the problem is to determine the position of a line of given length, and its directions, only, are known, it is obvious that the required line must lie somewhere in the line of the given directions. If the problem is to determine the position of a point, and the direction is given, but not the distance in the direction, the required point must be somewhere in the given line or direction. In either problem, it remains necessary definitely to locate only one point to fully determine the problem.

96. That which may be known in determining or locating a point, or a line, or other geometrical magnitude, when the necessary conditions are not all given, is not the position of the required point or magnitude, but the **locus**, or *place*, where one may expect to find it; thus, in § 95, the *locus* of the required point is the given line.

97. The **locus of a point** is the line, or lines, which include all those points, and only those points, that satisfy the given conditions in an indeterminate problem, some of the conditions necessary to determine the point being unknown.

98. The *locus* of a line or point is inclusive and exclusive; it not only *includes* all lines or points that satisfy the given conditions, but it *excludes* all lines or points that do not satisfy the given conditions.



99. Exercise. *Find the locus of a point that is equidistant from two given points.*

Join the given points ***B*** and ***C***, and bisect the line ***BC*** by the perpendicular ***EF***. ***EF*** is the required *locus*. Every point in ***EF*** is equidistant from ***B*** and ***C***. (PROP. XV.) Every point without ***EF*** is unequally distant from ***B*** and ***C***. (PROP. XV., COR. II.) It is, therefore, known that ***EF*** contains all the points that are equidistant from ***B*** and ***C***; that is, the bisecting perpendicular is the *locus* of the point that is equidistant from two given points.

SCIENCE OF GEOMETRY.

100. From all that precedes, it is seen that geometry is a **deductive science**, because its conclusions are logically drawn, or *deduced*, from that which is previously given or known.

The physical sciences are *inductive sciences*, because certain known facts lead to the assumption of a general theory, the truth of which is accepted in proportion as the deductions from the theory explain the given facts and other related facts.

101. That which is given in a specific theorem is called the **hypothesis**, or *condition;* and that which was previously known includes the definitions, axioms, postulates, and all preceding propositions, their corollaries and scholia; that which is given and that which is previously known together constitute the **premises.**

It is thus seen that the *definitions*, *axioms*, and *postulates* are the first premises, or the basis, of Geometry, as of other deductive sciences.

102. Proceeding from the *premises* by courses of reasoning called **demonstrations**, the conclusions are reached; and these, in turn, serve as premises for succeeding demonstrations. A **direct demonstration** is one that proceeds from the premises directly to the desired conclusion; an **indirect demonstration** is one that shows that some absurdity, something contrary to a known truth, must follow from supposing that the theorem is not true. The indirect demonstration is, therefore, called the *reductio ad absurdum.* **Demonstration by exhaustion** involves the *reductio ad absurdum;* it asserts that one of several alternatives must be true, and proves by the indirect method that all but one of these alternatives are false.

Thus, one quantity is either equal to another quantity, or it is greater or less than that quantity. If an absurdity follows the supposition that the one quantity is greater, and the supposition that it is less, than the other quantity, it must, by the method of exhaustion, be equal to it. Point out, in Book I., propositions that illustrate the different kinds of demonstrations.

103. The principal work required of the student in Geometry is demonstrating a given proposition or discovering a demonstration for a given theorem or problem. In *demonstrating*, the process is the **synthetic** process; that is, the

premises are *placed together* in such manner that the desired conclusion follows as a logical consequence. In *discovering a demonstration*, the process is the **analytic** process; that is, the desired conclusion is assumed to be true, and it is then examined with reference to the simpler propositions from which it may be deduced, until some such proposition is found which has for its conditions the conditions given in the problem.

Analysis thus leads from desired conclusions to known conditions. *Synthesis*, then, enables the student to present the demonstration by proceeding from the known conditions to the desired conclusions.

104. In certain books on Geometry, the authors proceed largely on the *principle of motion*. A point in motion is said to generate a line; a line in motion generates a surface; and a surface in motion generates a solid. The demonstrations, too, proceed on the same principle. Figures, by the motion of their parts, are supposed to assume successive positions, shapes, or forms. This involves the theory of limits. (See GENERAL SCHOLIA, BOOK III.)

In all books, however, it is considered possible to take up, or move to any position, any line, figure, plane, or solid, and to turn it in any direction.

BOOK II.—TRIANGLES.

SYNOPSIS.*

SECTION I.—EQUAL TRIANGLES.

105. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. I. If two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, they are equal in all their parts.

PROP. II. If two triangles have two angles and the included side of the one equal to two angles and the included side of the other, each to each, they are equal in all their parts.

PROP. III. If two triangles have three sides of the one equal to the three sides of the other, each to each, they are equal in all their parts.

PROP. IV. If two right-angled triangles have the hypotenuse and one side of the one equal to the hypotenuse and one side of the other, each to each, they are equal in all their parts.

PROP. V. If two right-angled triangles have the hypotenuse and one acute angle of the one equal to the hypotenuse and one angle of the other, each to each, they are equal in all their parts.

3. MISCELLANEOUS EXERCISES.

SECTION II.—EQUAL PARTS.

106. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. VI. In any isosceles triangle, the angles opposite the equal sides are equal.

PROP. VII. If a triangle is equilateral, it is also equiangular.

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* A statement of the theorems and problems is given at the beginning of the respective books, not only for purposes of review, but so that students may, if possible, be induced to make demonstrations of their own, without the assistance of the demonstrations in the text.

PROP. VIII. If two of the angles of a triangle are equal, the sides opposite them are equal, and the triangle is isosceles.

PROP. IX. If a triangle is equiangular, it is also equilateral.

PROP. X. Any exterior angle of a triangle is equal to the sum of the two opposite interior angles.

3. MISCELLANEOUS EXERCISES.

SECTION III.—UNEQUAL PARTS.

107. 1. DEFINITIONS. AXIOMS.

2. PROPOSITIONS.

PROP. XI. The sum of any two sides of a triangle is greater than the third side, and the difference between any two sides is less than the third side.

PROP. XII. The greater side in any triangle has the greater angle opposite to it.

PROP. XIII. The greater angle in any triangle has the greater side opposite to it.

PROP. XIV. If two triangles have two sides of the one equal to two sides of the other, each to each, but the included angles unequal, the third sides are unequal, and the triangle having the greater included angle has the greater third side.

PROP. XV. If two triangles have two sides of the one equal to two sides of the other, each to each, but the third sides unequal, the included angles are unequal, and the triangle having the greater third side has the greater included angle.

3. MISCELLANEOUS EXERCISES.

SECTION IV.—ANGLES.

108. 1. PROPOSITIONS.

PROP. XVI. The sum of the interior and exterior angles of any triangle, formed by extending the sides in the same order, is equal to six right angles.

PROP. XVII. The sum of the three angles of a triangle is equal to two right angles.

PROP. XVIII. The sum of any two angles of a triangle is equal to the supplement of the third angle; and the supplement of any one angle of a triangle, diminished by either of the others, equals the third angle.

PROP. XIX. The difference between any two angles of a triangle is twice the angle made by the line bisecting the third angle and the perpendicular from its vertex to the opposite side, or to the opposite side produced.

PROP. XX. The greater of any two angles of a triangle is equal to half of their sum plus half of their difference, and the less is equal to half of their sum minus half of their difference.

2. MISCELLANEOUS EXERCISES.

SECTION V.—CONCURRENT LINES.

109. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XXI. A straight line drawn from the middle point of one side of a triangle, parallel to a second side, and terminating in the third side, bisects the third side, and is equal to one-half of the second side.

PROP. XXII. The perpendiculars bisecting the sides of any triangle are concurrent lines.

PROP. XXIII. The straight lines bisecting the angles of any triangle are concurrent lines.

PROP. XXIV. The straight lines drawn from the vertices of any triangle, perpendicular to the opposite sides, are concurrent lines.

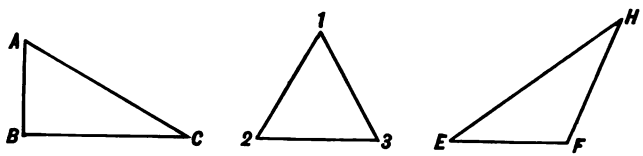
PROP. XXV. The medians, or straight lines drawn from the vertices to the middle points of the opposite sides of a triangle, are concurrent lines.

3. MISCELLANEOUS EXERCISES.

GENERAL SCHOLIA.

110. 1. INTERSECTION OF LOCI. 2. SYMMETRY.

SECTION I.—EQUAL TRIANGLES.



DEFINITIONS.

111. A **plane triangle** is a figure bounded by three straight lines.

The figure *A B C* is a *triangle;* it is bounded by the three straight lines *A B*, *B C*, and *A C*, and it has three angles, *A*, *B*, and *C*.

112. Triangles are named generally by the names of their three angles taken in any order, as the triangle *A B C*, *A C B*, *B A C*, *B C A*, etc.

113. A **right-angled triangle** is a triangle one of whose angles is a right angle. *A B C* is a *right-angled triangle.*

114. An **obtuse-angled triangle** is a triangle one of whose angles is an obtuse angle. *H F E* is an *obtuse-angled triangle.*

115. An **acute-angled triangle** is a triangle all of whose angles are acute angles. *1 2 3* is an *acute-angled triangle.*

116. The **hypotenuse** of a right-angled triangle is the side opposite the right angle, as *A C* in the triangle *A B C*.

117. The **base** of a triangle is the side upon which it is supposed to rest, as *B C* in the triangle *A B C*.

118. The sides and the angles of a triangle are called its parts ; thus, every triangle has six parts,—three sides and three angles.

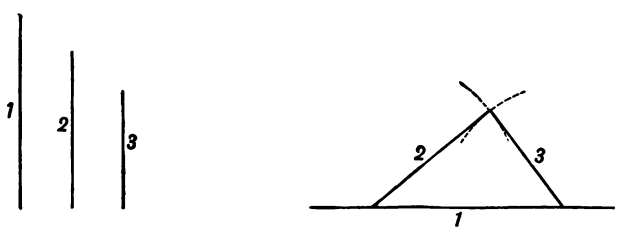
119. An angle in a triangle is said to be included between its two sides, and a side is said to be included between the two angles at its extremities.

The angle ***B*** in the triangle ***A B C*** is included between the sides ***B A*** and ***B C***, and the side ***B C***, between the angles ***B*** and ***C***.

120. Figures equal in all their parts, taken in the same order, are said to be *equal in all respects*, or they are said to be simply *equal;* when they are equal in area, they are said to be *equivalent.*

RULE AND COMPASS.

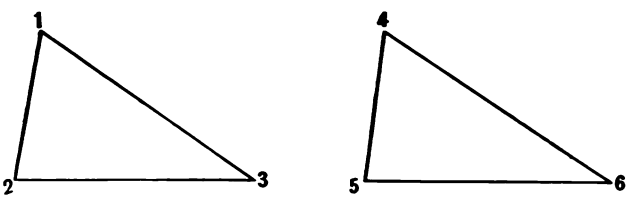
121. *To construct a triangle with the three sides given*



By means of the dividers, on any indefinite line take a part, *1*, equal to one of the given sides; with one of the extremities of this part as a center, and a distance between the dividers equal to a second side, draw an arc, as in the diagram; and from the other extremity draw an arc at a distance equal to the third side. Using the ruler, draw lines from the centers to the point where the arcs intersect. The triangle *1 2 3*, thus formed, has sides respectively equal to the given lines, and it is, therefore, the required triangle.

PROPOSITION I.

122. Theorem: ***If two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, they are equal in all their parts.***



Statement: Let the two triangles *1 2 3* and *4 5 6* have the side *1 2* equal to the side *4 5*, the side *1 3* equal to the side *4 6*, and the angle *1* equal to the angle *4*. The triangles are to be proved equal in all their parts.

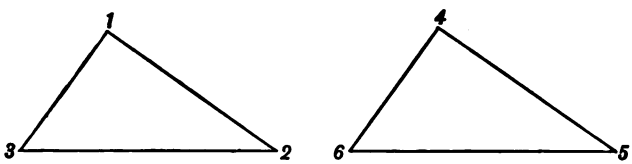
Construction: Place the triangle *1 2 3* upon the triangle *4 5 6* so that the point *1* shall fall upon the point *4*, and the line *1 2* shall take the direction of the line *4 5*.

Demonstration: Since *1 2* is equal to *4 5*, the point *2* will fall upon the point *5*; since the angle *1* is equal to the angle *4*, the line *1 3* will fall upon the line *4 6*; since *1 3* is equal to *4 6*, the point *3* will fall upon the point *6*. (BOOK I., PROP. IV.) Since the point *2* falls upon the point *5*, and the point *3* upon the point *6*, the line *2 3* coincides with *5 6*. (POST. I.) The two triangles have been made to coincide throughout, and they are, thus, proved equal in all their parts.

Conclusion: *1 2 3* and *4 5 6* being any two triangles, etc.

PROPOSITION II.

123. Theorem: ***If two triangles have two angles and the included side of the one equal to two angles and the included side of the other, each to each, they are equal in all their parts.***



Statement: Let the two triangles *1 2 3* and *4 5 6* have the angle *3* equal to the angle *6*, the angle *2* equal to the angle *5*, and the included side *2 3* equal to the included side *5 6*. Prove that the triangles are equal in all their parts.

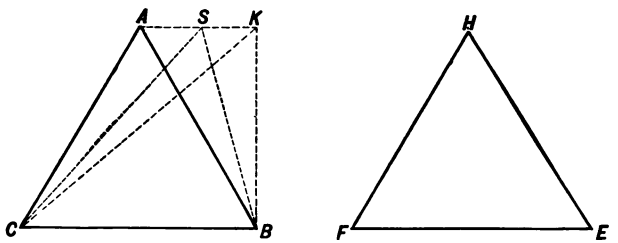
Construction: Place the triangle *1 2 3* upon the triangle *4 5 6*, so that the point *3* shall fall on *6*, and the line *3 2* on *6 5*.

Demonstration: Since the angle *3* is equal to the angle *6*, the side *3 1* will take the direction of the side *6 4*; since *3 2* is equal to *6 5*, the point *2* will fall upon the point *5*; since the angle *2* is equal to the angle *5*, the side *2 1* will take the direction of the side *5 4*. (BOOK I., PROP. V.) As *3 1* takes the direction of *6 4*, the point *1* is somewhere in the line *6 4*; as *2 1* takes the direction of the side *5 4*, the point *1* is somewhere in the line *5 4*. The point *1*, being at the same time on the lines *6 4* and *5 4*, it is at *4*, their point of intersection. (BOOK I., PROP. V., COR. IV.) The two triangles coincide throughout, and they are, thus, proved equal.

Conclusion: *1 2 3* and *4 5 6* being any two triangles, etc.

PROPOSITION III.

124. Theorem: ***If two triangles have three sides of the one equal to the three sides of the other, each to each, they are equal in all their parts.***



Statement: Let the two triangles *A C B* and *H F E* have the side *A C* equal to *H F*, the side *A B* equal to *H E*, and *B C* equal to *E F*. The triangles are to be proved equal in all their parts.

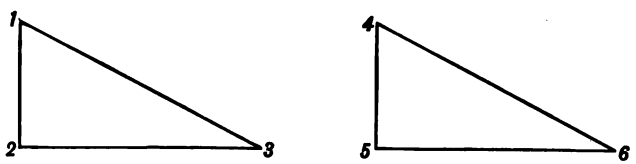
Construction: Place the triangle *H F E* upon the triangle *A C B*, so that the point *F* shall fall on *C*, and the line *F E* on *C B*. Since *C B* is equal to *F E*, the point *E* will fall on *B*. If *H* does not fall on *A*, it will fall at some other point, as *K*. Draw *A K*, and to its middle point, draw *C S* and *B S*.

Demonstration: *A C* and *K C* are equal, as each is equal to *H F*. *C S* is, then, perpendicular to *A K*. (BOOK I., PROP. XV., COR. III.) Similarly, *A B* and *K B* being equal, *B S* is, also, perpendicular to *A K*. This is impossible. (BOOK I., PROP. XI.) The supposition that *H* does not fall on *A* must be abandoned. The point *H* falls on *A*. The side *H F*, then, coincides with *A C*, and *H E*, with *A B*. (POST. I.) The two triangles coincide throughout and are, thus, proved equal.

Conclusion: *A B C* and *H E F* being any two triangles, etc.

PROPOSITION IV.

125. Theorem: ***If two right-angled triangles have the hypotenuse and one side of the one equal to the hypotenuse and one side of the other, each to each, they are equal in all their parts.***



Statement: Let the triangles *1 2 3* and *4 5 6* be right-angled triangles, with right angles at *2* and *5*. Let the hypotenuse *1 3* equal the hypotenuse *4 6*, and the side *1 2* equal the side *4 5*. The two triangles are to be proved equal.

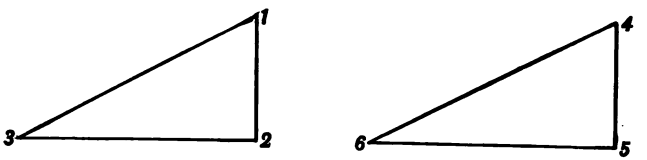
Construction: Place the triangle *4 5 6* upon the triangle *1 2 3*, so that the point *5* shall fall upon the point *2*, and so that the line *5 6* shall take the direction of the line *2 3*.

Demonstration: Since the angle *5* is equal to the angle *2*, the line *5 4* will coincide in direction with the line *2 1*. Since *5 4* is equal to *2 1*, the point *4* will fall upon the point *1*. (BOOK I., PROP. IV.) Since *4 6* and *1 3* are equal oblique lines, they meet *2 3* at the same distance from *2*, the foot of the perpendicular; that is, the point *6* will fall upon the point *3*. (BOOK I., PROP. XXIV.) The two triangles have thus been made to coincide throughout, and they are, thus, proved equal.

Conclusion: *1 2 3* and *4 5 6* being any two right-angled triangles, etc.

PROPOSITION V.

126. Theorem: ***If two right-angled triangles have the hypotenuse and one acute angle of the one equal to the hypotenuse and one angle of the other, each to each, they are equal in all their parts.***



Statement: Let the triangles *1 2 3* and *4 5 6* have right angles at *2* and *5*. Let the hypotenuse *1 3* equal the hypotenuse *4 6*, and the angle *3* equal the angle *6*. The two triangles are then equal in all their parts.

Construction: Place the triangle *4 5 6* upon the triangle *1 2 3* so that the point *6* shall fall upon the point *3*, and so that the line *6 5* shall take the direction of the line *3 2*, the point *5* falling somewhere on the line *3 2*.

Demonstration: Since the angle *6* is equal to the angle *3*, the line *6 4* will take the direction of the line *3 1*. Since *6 4* is equal to *3 1*, the point *4* will fall upon the point *1*. (BOOK I., PROP. IV.) *4 5* and *1 2* are perpendiculars, and must, therefore, coincide ; for, from a given point, only one perpendicular can be drawn to a given straight line. (BOOK I., PROP. XII.) The two triangles, then, coincide throughout, and they are, thus, proved equal.

Conclusion: *1 2 3* and *4 5 6* being any two right-angled triangles, etc.

Scholium I. Propositions I. to V. are more frequently used than any other propositions in Geometry. The following corollaries may be used simply as exercises. In the two following exercises, assume that all the propositions establishing the equality of triangles have been proved.

Corollary I. Any two triangles, with two angles and a side opposite one of these angles in the one equal, each to each, to two angles and the side opposite the corresponding angle in the other, are equal in all their parts. (Superposition and BOOK I., PROP. XXV., COR. V.)

Corollary II. Two triangles, with two sides and the angle opposite the greater in the one, respectively equal to two sides and the angle opposite the greater in the other, are equal in all their parts. (Superposition and BOOK I., PROP. XXIV., COR. I.)

Corollary III. Two triangles, with two sides and one of two acute angles opposite these sides in the one, respectively equal to two sides and the corresponding one of two acute angles opposite these sides in the other, are equal in all their parts. (Same.)

Corollary IV. Two triangles, with two sides of the one respectively equal to two sides of the other, the two angles opposite these sides, in each triangle, being both acute, or one acute and the other obtuse, and one of these angles in the one being equal to the corresponding angle in the other, are equal in all their parts. (Same.)

Exercises. 1. By means of triangles, prove Propositions XII., XV., and XVI. in Book I.

2. By triangles, prove Props. XXI. to XXV. in Book I.

MISCELLANEOUS EXERCISES.*

THEOREMS.

127. Prove the following:

I. 1. Parallel straight lines are everywhere equally distant.

2. Every point in the bisectrix of an angle is equally distant from the sides of the angle.

3. Two equal straight lines drawn from a point to a straight line make equal angles with that line.

II. 4. If two straight lines bisect each other, the lines joining opposite extremities are parallels.

5. If through the middle point of a straight line, terminating in two parallel lines, a second straight line be drawn, also terminating in the parallels, this second straight line will also be bisected at that point.

6. The middle point of that part of a transversal intercepted between two parallels is equidistant from those parallels.

III. 7. Perpendiculars from the extremities of a line to any line which bisects it are equal.

8. The portions of two parallel straight lines intercepted by two transversals, drawn through a point midway between the parallels, are equal.

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* If desired, all sets of exercises marked II. may be reserved for review; and those marked III., for subsequent review, or examinations.

PROBLEMS.

128. Lines and Points. I. 9. Bisect any angle.

10. Bisect a finite straight line.

II. 11. Find the fourth part of any angle.

12. In a given straight line, find a point equidistant from two given lines that intersect. When is this impossible?

III. 13. Draw a line which shall pass through a given point, and make equal angles with two given intersecting lines.

14. On the same side of a straight line, from two points given in position, draw lines that shall meet in the given line and make equal angles with it.

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129. Constructions.* Construct triangles, the following parts being given:

I. 15. The three sides.

16. The three angles.

17. Two sides and the included angle.

18. Two angles and the included side.

19. The hypotenuse and one side of a right-angled triangle.

20. The hypotenuse and one angle of a right-angled triangle.

II. 21. Two sides and the angle opposite one.

* Two of the required constructions are indeterminate. Discuss the conditions that make them indeterminate.

22. Two angles, and a side opposite one.

III. 23. Two sides, and the angle opposite the greater.

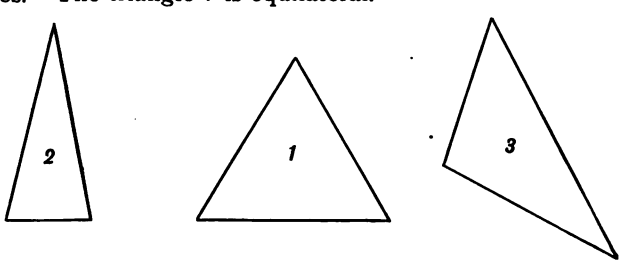
24. Two sides, and one of two acute angles opposite these sides.

25. Two sides, and one of the two angles opposite these sides, the angle opposite the other side being referred to as acute or obtuse.

SECTION II.—EQUAL PARTS.

DEFINITIONS.

130. An **equilateral triangle** is a triangle with three equal sides. The triangle *1* is equilateral.



131. An **equiangular triangle** is a triangle with three equal angles. The triangle *1* is equiangular.

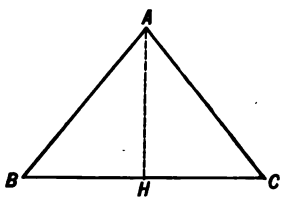
132. An **isosceles triangle** is a triangle with two of its sides equal, as the triangle *2*.

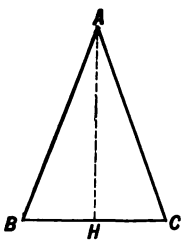
133. A **scalene triangle** is a triangle with none of its sides equal, as the triangle *3*.

NOTE.—Triangles are classified with respect to their angles, as *right*, *obtuse*, or *acute*; with respect to their sides, as *equilateral*, *isosceles*, or *scalene*.

PROPOSITION VI.

134. Theorem: ***In any isosceles triangle, the angles opposite the equal sides are equal.***





Statement: Let the triangle *A B C* be isosceles; that is, let two of the sides, as *A B* and *A C*, be equal to each other. The angles *B* and *C*, opposite the equal sides, are equal.

Construction: From the point *A* draw the straight line *A H* to the middle point of *B C*.

Demonstration: In the two triangles *A B H* and *A C H*, *B H* is equal to *H C*, by construction; *A H* is common, by construction; and *A B* is equal to *A C*, by hypothesis. The two triangles have then the three sides of the one equal to the three sides of the other, each to each, and they are, therefore, equal. (PROP. III.) The corresponding parts being equal to each other, the angle *B* is equal to the angle *C*.

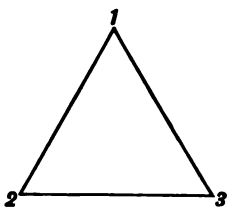
Conclusion: *A B C* being any isosceles triangle, it follows, etc.

Exercises. 1. Prove the proposition on this page by reference to oblique lines.

2. How many degrees are there in an angle whose supplement is seven times the complement?

PROPOSITION VII.

135. Theorem: ***If a triangle is equilateral, it is also equiangular.***



Statement: Let the triangle *1 2 3* be equilateral; that is, let the three sides *1 3*, *3 2*, and *2 1* be equal to one another. Then will the three angles *1*, *2*, and *3* be equal to one another.

Demonstration: Since the side *1 2* is equal to the side *1 3*, the angle *3* is equal to the angle *2*. (PROP. VI.) Since the side *3 2* is equal to the side *1 3*, the angle *1* is equal to the angle *2*. (PROP. VI.) The angles, *3* and *1*, being each equal to the angle *2*, are equal to each other. (AXIOM I.) The three angles, *1*, *2*, and *3*, are then equal to one another. The triangle is, therefore, equiangular.

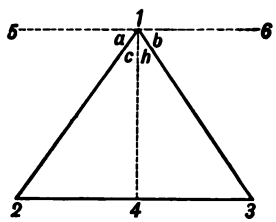
Conclusion: The triangle *1 2 3* being any equilateral, etc.

Exercises. 1. Prove any two angles of an equilateral triangle equal by drawing a line from the vertex of the third angle to the middle point of the base.

2. The difference between two supplementary angles is 50°. What are the angles?

PROPOSITION VIII.

136. Theorem: ***If two of the angles of a triangle are equal, the sides opposite them are equal, and the triangle is isosceles.***



Statement: Let the triangle *1 2 3* have the angle *2* equal to the angle *3*. The side *1 3* is then equal to the side *1 2*.

Construction: Through the vertex *1*, draw *5 6* parallel to *2 3*, and *1 4* perpendicular to *5 6*.

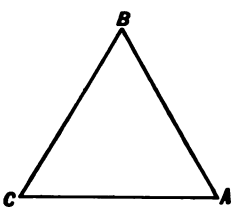
Demonstration: The angles *2* and *3* being equal, their alternate angles *a* and *b* are equal. (BOOK I., PROP. XVI.) As *a* and *b* are equal, their complements *c* and *h* are equal. (AXIOM IV.) The angles at *4* are equal, as they are right angles; for, if *1 4* is perpendicular to *5 6*, it is also perpendicular to *2 3*. (BOOK I., PROP. XIV.)

The triangles *1 4 2* and *1 4 3* have the angle *c* equal to the angle *h*, the side *1 4* common, and the angles at *4* equal. They are, therefore, equal. (PROP. II.) The sides *1 2* and *1 3*, opposite the equal angles, are, then, equal to each other.

Conclusion: *1 2 3* being any triangle, with two of the angles equal, it follows, in general, that: If two of the angles, etc.

Proposition IX.

137. Theorem: ***If a triangle is equiangular, it is also equilateral.***



Statement: Let the triangle *B A C* be equiangular; that is, let the three angles *B*, *A*, and *C* be equal to one another. The three sides *B A*, *A C*, and *C B* are also equal to one another.

Demonstration: Since the angle *A* is equal to the angle *C*, the side *B A* is equal to the side *B C*. (Prop. VIII.) Since the angle *B* is equal to the angle *C*, the side *A C* is equal to the side *B A*. (Prop. VIII.)

The sides *B C* and *A C* being each equal to the side *B A*, they are equal to each other. (Axiom I.) The three sides *B A*, *A C*, and *C B* are thus shown to be equal to one another, and the triangle is equilateral.

Conclusion: *A B C* being any equiangular triangle, etc.

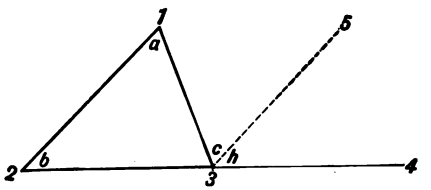
Exercises. 1. Prove Proposition VIII. by drawing a line from the third angle to the middle of the opposite side.

2. Prove Proposition VIII. by drawing a line from the third angle perpendicular to the opposite side.

3. One of two supplementary angles is five times the other. *What are the angles?*

PROPOSITION X.

138. Theorem: ***Any exterior angle of a triangle is equal to the sum of the two opposite interior angles.***



Statement: Let *1 2 3* be a triangle, and let one of the sides, as *2 3*, be extended, making the exterior angle *1 3 4*. The angle *1 3 4* is to be proved equal to the sum of the opposite interior angles, *a* and *b*.

Construction: Through *3*, the vertex of the exterior angle, draw the line *3 5* parallel to *2 1*.

Demonstration: The exterior angle is equal to the sum of the angles *c* and *h*, because the whole is equal to the sum of all its parts. (AXIOM II.) The angle *c*, however, is equal to the angle *a*, as they are alternate angles (BOOK I., PROP. XVI.); and the angle *h* is equal to the angle *b*, as they are opposite exterior and interior angles on the same side. (BOOK I., PROP. XVII.)

The sum of the angles *c* and *h*, or the exterior angle *1 3 4*, is, therefore, equal to the sum of the two opposite interior angles *a* and *b*. (AXIOM III.)

Conclusion: *1 2 3* being any triangle, and *1 3 4* being any one of its exterior angles, it follows, in general, etc.

Corollary I. The sum of the exterior angles of any triangle, made by producing its sides in the same order, is four right angles.

Corollary II. The sum of the exterior angles of any rectilinear figure is four right angles.

MISCELLANEOUS EXERCISES.

THEOREMS.

139. Prove the following: I. 1. A straight line cutting the sides of an isosceles triangle, and parallel to the base, makes equal angles with the sides.

2. The straight line which bisects a vertical exterior angle of an isosceles triangle is parallel to the base

3. If the bisector of the exterior angle at the vertex of a triangle is parallel to the base, the triangle is isosceles.

4. If the two exterior angles at the base of a triangle are equal, the triangle is an isosceles triangle.

5. The perpendicular from the vertical angle of an isosceles triangle to the base bisects the base.

6. The line from the vertex to the middle point of the base of an isosceles triangle is perpendicular to the base.

7. If the perpendicular from the vertex to the base of a triangle bisects the base, the triangle is isosceles.

8. The base of an isosceles triangle forms a second isosceles triangle with the lines that bisect the angles at the base.

9. The line that bisects the angle at the vertex of an isosceles triangle bisects the base.

10. A straight line drawn from any point in the bisectrix of an angle to either side, parallel to the other side, makes with the bisectrix and the side it meets an isosceles triangle.

II. 11. Points in the sides of an isosceles triangle equidistant from the base are equidistant from the vertex.

12. If the angles at the base of an isosceles triangle are bisected, the line which joins their intersection with the vertical angle bisects the vertical angle.

13. The perpendiculars from the extremities of the base of an isosceles triangle to the opposite sides are equal.

14. If the perpendiculars from two angles of a triangle to the opposite sides are equal, the triangle is an isosceles triangle.

III. 15. The lines joining the extremities of the base of an isosceles triangle to the middle points of the opposite sides are equal.

16. The triangles made with the base of an isosceles triangle by the lines bisecting the angles at the base are equal in all respects.

17. The line joining the vertices of two isosceles triangles on opposite sides of the same base bisects the base at right angles.

18. The lines joining the extremities of bases of two opposite or vertical isosceles triangles are equal.

19. If two lines drawn from vertices of a triangle make equal angles with the opposite sides, the triangle is an isosceles triangle.

PROBLEMS.

140. Lines and Points. I. 20. From two given points to draw two equal straight lines which shall meet in the same point of a line given in position.

II. 21. From a given point on a line, erect a perpendicular to the line.

22. From a given point without a line, draw a perpendicular to the line.

III. 23. Trisect a given finite straight line. (Equilateral triangle.)

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141. Constructions. Construct triangles, the following parts being given :

I. 24. The base, and one of two equal sides.

25. The base, and the sum of two equal sides.

II. 26. The base, and the sum of one of the equal sides and the perpendicular from the vertex to the base.

27. The base, and one of two equal angles at the base.

III. 28. The base, and the vertical angle of an isosceles triangle.

29. The base, an angle at the base, and the difference of the two other sides.

30. The base, an angle at the base, and the sum of the two other sides.

SECTION III.—UNEQUAL PARTS.

DEFINITIONS.

142. An **inequality** is an expression of the relative size of two quantities that are not equal.

143. The **sign** of **inequality** is $<$ or $>$; it is written with the opening toward the greater quantity. If the less quantity is written first, the sign $<$ is used, and it is read *less than ;* if the greater quantity is written first, the sign $>$ is used, and it is read *greater than.*

$a > b$ is read, a is *greater than* b; $a < b$ is read, a is *less than* b.

AXIOMS.*

144. Axiom IX. If equals are added to unequals, the results are unequals.

Axiom X. If equals are subtracted from unequals, the results are unequals.

Axiom XI. If equals are multiplied by unequals, the results are unequals.

Axiom XII. If equals are divided by unequals, the results are unequals.

Exercise. State four other axioms as to unequals and equals.

Unequals subtracted from equals give an inverted inequality. The results to which reference is made in the preceding axioms are not inverted inequalities.

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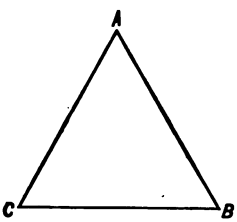
* For other axioms, see p. 6.

PROPOSITION XI.

145. Theorem: ***The sum of any two sides of a triangle is greater than the third side, and the difference between any two sides is less than the third side.***



Statement: Let ABC be a triangle. The sum of any two sides, as AC and AB, is greater than CB, the third side; and the difference between any two sides, as AC and AB, is less than CB, the third side.

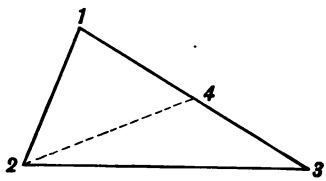
Demonstration: As a straight line is the shortest distance between two points, the distance between C and B, measured on the broken line CA, AB, is greater than the distance between the same two points measured on the straight line CB. (POST. II.) Therefore, $AC + AB > CB$. In similar manner, it is proved, that $AB + CB > AC$, which is the same as, $AC < AB + CB$.

From the inequality, $AC < AB + CB$, subtract $AB = AB$; and there remains the inequality, $AC - AB < CB$; (AXIOM X.) that is, the difference between AC and AB is less than CB.

Conclusion: ABC being any triangle, and AB and AC being any two sides, it follows, in general, that: The sum of any two sides of a triangle is greater than the third side, and the difference between any two sides is less than the third side.

PROPOSITION XII.

146. Theorem: ***The greater side in any triangle has the greater angle opposite to it.***



Statement: In the triangle *1 2 3*, let the side *1 3* be greater than the side *1 2*. Prove that the angle *2*, opposite the greater side *1 3*, is greater than the angle *3*, opposite the lesser side *1 2*.

Construction: Lay off on the greater side, *1 3*, a part of it, *1 4*, equal to the lesser side, *1 2*. Draw the line *2 4*.

Demonstration: The angle *1 2 3* is greater than the angle *1 2 4*, for the whole is greater than any of its parts. (AXIOM II.) The angle *1 2 4* is, however, equal to the angle *1 4 2*, as they are opposite equal sides (PROP. VI.); and the angle *1 4 2*, as it is exterior to the triangle *2 4 3*, is greater than the angle *1 3 2*. (PROP. X.) It is, thus, proved, that

$$1\,2\,3 > 1\,2\,4 = 1\,4\,2 > 1\,3\,2.$$

As the angles *1 2 4* and *1 4 2* are equal, there results the continued inequality:

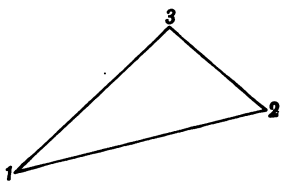
$$1\,2\,3 > 1\,4\,2 > 1\,3\,2;$$ (AXIOM I.)

showing that the angle *1 2 3* is greater than the angle *1 3 2*.

Conclusion: *1 2 3* being any triangle, etc.

Proposition XIII.

147. Theorem: ***The greater angle in any triangle has the greater side opposite to it.***



Statement: Let *1 3 2* be a triangle, with the angle *2* greater than the angle *1*. Prove that the side *1 3*, opposite *2*, is greater than the side *3 2*, opposite *1*.

Demonstration: The side *1 3* is not equal to the side *3 2*, for then the angle *2* must be equal to the angle *1* (Prop. vi.); but this is contrary to the hypothesis. The side *1 3* is not less than the side *3 2*, for then the angle *2* must be less than the angle *1* (Prop. xii.); but this is contrary to the hypothesis.

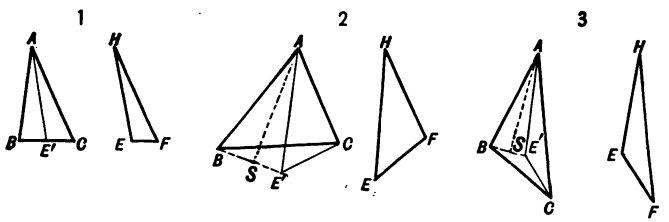
As the side *1 3*, opposite the greater angle, is neither equal to nor less than the side *3 2*, which is opposite the lesser angle, it must be greater than *3 2*.

Conclusion: *1 3 2* being any triangle, and *2* being any angle greater than *1*, any other angle, it follows, in general, that: The greater angle in any triangle has the greater side opposite to it.

Exercise. Prove that the two straight lines drawn from any point within a triangle to the extremities of any side, are together less than the two other sides of the triangle, but include a greater angle.

PROPOSITION XIV.

148. Theorem: ***If two triangles have two sides of the one equal to two sides of the other, each to each, but the included angles unequal, the third sides are unequal, and the triangle having the greater included angle has the greater third side.***



Statement: Let *A B C* and *H E F* have *A B* and *A C* respectively equal to *H E* and *H F*; but the angle *A* greater than the angle *H*. Prove the third side *B C* greater than the third side *E F*.

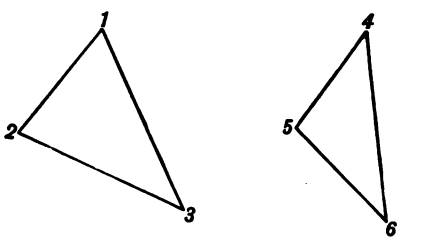
Construction: Place the triangle *H E F* upon the triangle *A B C*, so as to take the position *A E' C*. The angle *H* being less than the angle *A*, the line *H E* falls between *A B* and *A C*. If *E'* does not fall on *B C*, draw *B E'*; and, then, draw *A S* to the middle point of *B E'*.

Demonstration: When *E* falls *on B C*, *B C* is greater than *E' C*, or its equal, *E F*, *E' C* being part of *B C*. (AXIOM II.) When *E* does *not* fall *on B C*, as *A S* is the bisecting perpendicular to *B E'*, *C* and *E'* being on the same side, the oblique line, the base *B C*, is greater than the base *E' C*, or its equal, *E F*. (BOOK I., PROP. XV., CORS. III. and I.)

Conclusion: *A B C* and *H E F* being any two triangles, etc.

PROPOSITION XV.

149. Theorem: ***If two triangles have two sides of the one equal to two sides of the other, each to each, but the third sides unequal, the included angles are unequal, and the triangle having the greater third side has the greater included angle.***



Statement: Let the triangles *1 2 3* and *4 5 6* have the sides *1 2* and *1 3* respectively equal to the sides *4 5* and *4 6*, but the third side *2 3* greater than the third side *5 6*. Prove that the angle *1* is greater than the angle *4*.

Demonstration: The angle *1* cannot be equal to the angle *4*, for then the side *2 3* would be equal to the side *5 6* (PROP. I.); but this is contrary to the hypothesis. The angle *1* cannot be less than the angle *4*, for then the side *2 3* would be less than the side *5 6* (PROP. XIV.), which is also contrary to the hypothesis.

As the angle *1* is neither equal to, nor less than, the angle *4*, it must be greater than the angle *4*.

Conclusion: *1 2 3* and *4 5 6* being any two triangles, with two sides of the one equal to two sides of the other, etc.

MISCELLANEOUS EXERCISES.

THEOREMS.

150. Prove the following: I. 1. Any side of a triangle is greater than the difference between the two other sides.

2. A straight line drawn from the vertex of an isosceles triangle to the base produced, is longer than either of the equal sides.

3. If the angles at the base of any triangle are bisected, the line bisecting the smaller angle is the longer line.

4. The three lines drawn from any point within a triangle to the three vertices are together greater than half the sum of the three sides.

5. The three lines drawn from any point within a triangle to the three vertices are together less than the sum of the three sides.

II. 6. A straight line drawn from the vertex of a triangle to the middle point of the base, makes with the base an obtuse or an acute angle, according as the side opposite is the greater or the less of the two opposite sides.

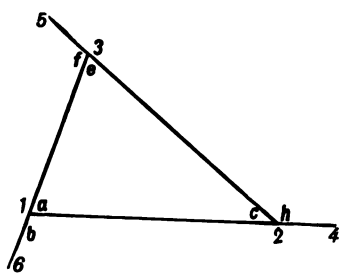
7. If one side of a triangle is greater than the other, any line from the included angle to the base is less than the longer side.

III. 8. If from two given points two straight lines be drawn to the same point in a given straight line, and equally inclined to this line, the sum of these two lines will be less than the sum of the two lines drawn from the two given points to any other point of the given line.

SECTION IV.—ANGLES.

PROPOSITION XVI.

151. Theorem: ***The sum of the interior and exterior angles of any triangle, formed by extending the sides in the same order, is equal to six right angles.***



Statement: Let *1 2 3* be any triangle, with the sides *1 2*, *2 3*, and *3 1* extended respectively to *4*, *5*, and *6*, making the three exterior angles, *h*, *f*, and *b*. The three interior, together with the three exterior, angles are equal to six right angles.

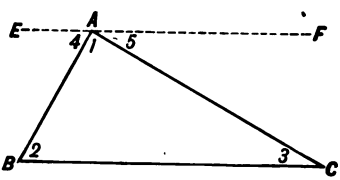
Demonstration: The triangle has three vertices, at each one of which there is a set of angles consisting of an interior and an exterior angle. The angles of each set, being adjacent, are equal to two right angles (BOOK I., PROP. VI.); and the angles of the three sets are, therefore, equal to three times two right angles, or six right angles.

Conclusion: *1 2 3* being any triangle, it follows, etc.

Corollary. The interior and exterior angles of any polygon equal two right angles multiplied by the number of sides.

PROPOSITION XVII.

152. Theorem: ***The sum of the three angles of a triangle is equal to two right angles.***



Statement: Let *A B C* be any triangle. The sum of *1*, *2*, and *3*, its three angles, is equal to two right angles.

Construction: Through *A*, draw *E F* parallel to *B C*.

Demonstration: In the figure there are two sets of angles —the three angles of the triangle, and the three angles at *A*. The angle *2* of the one set is equal to its alternate angle, the angle *4* of the other set. (BOOK I., PROP. XVI.) For a similar reason, the angle *3* of the one set is equal to the angle *5* of the other set. The angle *1* is common to both sets. The three angles of the one set are, then, equal to the three angles of the other set. (AXIOM III.)

The three angles at *A* are, however, equal to two right angles. (BOOK I., PROP. VI., COR. I.) The three angles of the triangle are, therefore, also equal to two right angles.

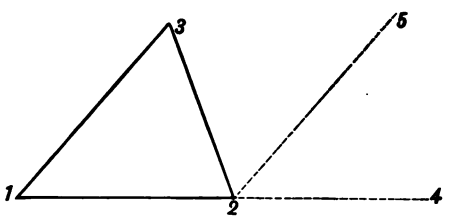
Conclusion: The triangle *A B C* being any triangle, etc.

Corollary I. The interior angles of any polygon equal two right angles multiplied by the number of sides, less two.

Corollary II. In a right-angled triangle, each acute angle is the complement of the other.

PROPOSITION XVIII.

153. Theorem: ***The sum of any two angles of a triangle is equal to the supplement of the third angle; and the supplement of any one angle of a triangle, diminished by either of the others, equals the third angle.***



Statement: Let *1 2 3* be any triangle. The sum of *1* and *3* equals the exterior angle which is the supplement of *2;* and the supplement of *2* diminished by *1* equals *3.*

Construction: Extend *1 2* to *4.* Draw *2 5* parallel to *1 3.*

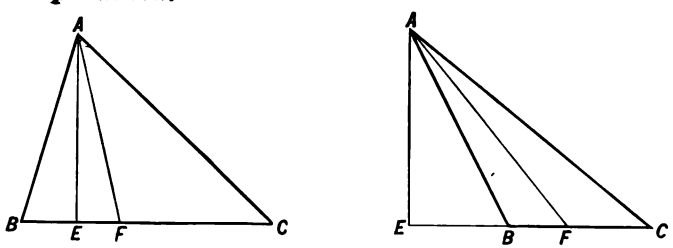
Demonstration: The angles *1* and *3* are the supplement of the angle *2,* as the three angles equal two right angles. (PROP. XVII.) The angle *3 2 4* is the supplement of *2,* its adjacent angle. (BOOK I., PROP. VI.) The angles *1* and *3* are, therefore, together equal to the exterior angle *3 2 4,* the supplement of the angle *2.*

From *3 2 4* subtract *3 2 5,* equal to *3,* its alternate angle (BOOK I., PROP. XVI.), and there will remain *5 2 4,* equal to *1,* the opposite interior angle on the same side. (BOOK I., PROP. XVII.)

Conclusion: The triangle *1 2 3* being any triangle, etc.

PROPOSITION XIX.

154. Theorem: ***The difference between any two angles of a triangle is twice the angle made by the line bisecting the third angle and the perpendicular from its vertex to the opposite side, or to the opposite side produced.***



Statement: Let ABC be any triangle, B and C being any two of its angles. The difference between B and C is twice the angle EAF, made by AF, bisecting the third angle A, and AE, drawn from A perpendicular to BC, or BC produced.

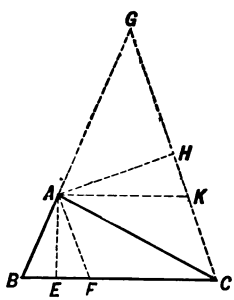
Demonstration: The angle $B = AFC - BAF$ (PROP. XVIII.);
and $C = AFB - FAC$;
but, $BAF = FAC$, by hypothesis;
subtracting, then, $B - C = AFC - AFB$, but,
the angle $AFC = E + EAF$; and (PROP. XVIII.)
$AFB = E - EAF$; (PROP. XVII., COR.)
subtracting, $AFC - AFB = 2\,EAF$; (AXIOM IV.)
therefore, $B - C = 2\,EAF$. (AXIOM I.)

Conclusion: As ABC is any triangle, it follows, etc.

Exercises. 1. $A = 35°$; $C = 55°$; $B = ?$ 2. $C = 63°$; $A = 45°$; supplement of $B = ?$ 3. In isosceles triangle, $A = 48°$; others?

PROPOSITION XX.

155. Theorem: ***The greater of any two angles of a triangle is equal to half of their sum plus half of their difference; and, the less is equal to half of their sum minus half of their difference.***



Statement: Let *A B C* be any triangle; and let the angle *B* be greater than the angle *C*. The angle *B* is equal to half of the sum of *B* and *C*, plus half of their difference, and the angle *C* is equal to half of the sum minus half of the difference.

Construction: Extend *B A* to *G*, making *A G* equal to *A C*. Draw *G C*. Draw *A E* perpendicular to *B C*, and *A F* bisecting the angle *A*. Draw *A K* perpendicular to *A E*; and, *A H* perpendicular to *A F*.

Demonstration: *B A F* and *C A F* are equal, by construction; but *B A F* is the complement of *G A H*, since *H A F* is a right angle (BOOK I., PROP. VI., COR. I.); and *C A F* is the complement of *C A H*, by construction. *G A H* and *C A H* are, therefore, equal. (AXIOM IV.) Together, *G A H* and *C A H* are

equal to the sum of the opposite interior angles *B* and *C*. (PROP. XVIII.) Each one is, therefore, equal to half of the sum of *B* and *C*. (AXIOM VI.)

HAK and *FAE* are equal, as each is the complement of *KAF*. (AXIOM IV.) Each of these angles is, then, equal to half of the *difference* of *B* and *C*. (PROP. XIX.)

AK is parallel to *BC*, as each is perpendicular to *AE*. (BOOK I., PROP. XIII.) The angle *B*, then, is equal to the opposite exterior angle *GAK* (BOOK I., PROP. XVII.), which is equal to *GAH* plus *HAK*, or half of the sum plus half of the difference of *B* and *C*. The angle *C* is equal to the alternate angle *KAC*, which is equal to *CAH* minus *HAK*, or half of the sum minus half of the difference of *B* and *C*.

Conclusion: *ABC* being any triangle, it follows, etc.

MISCELLANEOUS EXERCISES.

THEOREMS.

156. Prove the following: I. 1. The hypotenuse is greater than either of the other sides of a right-angled triangle.

2. The exterior angles at the base of any triangle are together greater than two right angles.

3. The perpendicular from the right angle to the hypotenuse of a right-angled triangle divides the triangle into two triangles, which are mutually equiangular with each other and with the right-angled triangle.

4. Prove that the three angles of a triangle are equal to two right angles by joining any vertex with any point in the opposite side.

5. Prove that the three angles of a triangle are equal to two right angles by extending one of the sides.

6. The sum of the exterior angles at the base of any triangle is equal to two right angles, plus the angle at the vertex of the triangle.

7. Either exterior angle at the base of an isosceles triangle is equal to the angle contained by the bisectors of the angles at the base.

II. 8. If, in a right-angled triangle, one of the acute angles is one-third of a right angle, the side opposite to this acute angle is one-half of the hypotenuse.

9. Two sides of one triangle are respectively equal and parallel to two sides of another triangle. Prove that the third sides are equal and parallel.

10. If through any point in a line which meets another line, at any point not its extremity, a line be drawn parallel to the other line, the bisectors of the adjacent angles will intercept equal portions of the line thus drawn.

11. The bisectors of the angles at the base of any triangle include an angle which is equal to the vertical angle and half of the angles at the base.

12. The line which joins points in the side of an isosceles triangle, equally distant from the extremities of the base, is parallel to the base.

III. 13. Lines bisecting the two interior angles on the same side of the transversal to two parallel straight lines, meet at right angles.

14. Any angle of a triangle is obtuse, acute, or a right angle, if the line from its vertex to the middle point of the opposite side is respectively less than, greater than, or equal to, half of the opposite side.

15. A straight line bisecting the vertical angle of a triangle divides the base into two segments, or parts, each of which is less than the adjacent side.

16. The line joining one extremity of the base of an isosceles triangle with the extremity of the opposite side, produced to double its length, is perpendicular to the base.

PROBLEMS.

157. Lines and points. I. 17. Draw a line parallel to the base of any triangle equal to the sum of the segments, or parts, of the other sides intercepted between the base and the required line.

II. 18. In an isosceles triangle, draw a line parallel to the base, and equal to each of the parts of the two sides included between the required line and the base.

III. 19. Trisect a right angle.

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158. Constructions. Construct triangles, the following parts being given:

I. 20. Hypotenuse and sum of the other sides of a right-angled triangle.

21. Hypotenuse and difference between the other sides of a right-angled triangle.

II. 22. The sum of the three sides, or perimeter, and the angles at the base.

23. The perimeter, and one acute angle.

III. 24. The sum of the hypotenuse and one side of an isosceles right-angled triangle.

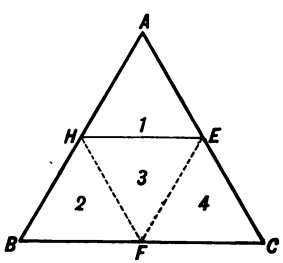
SECTION V.—CONCURRENT LINES.*

DEFINITION.

159. If several lines *run together*, or meet, at one point, they are said to be **concurrent lines.**

PROPOSITION XXI.

160. Theorem: ***A straight line drawn from the middle point of one side of a triangle, parallel to a second side, and terminating in the third side, bisects the third side, and is equal to one-half of the second side.***



Statement: Let *H* be the middle point of *A B*, any side of the triangle *A B C*. Let *H E* be parallel to *B C* and terminate in *A C*. The straight line *H E* bisects *A C* and is equal to one-half of *B C*.

Construction: From *H* draw *H F* parallel to *A C*. Join *F* and *E* by the straight line *F E*.

* The five following propositions may be reserved for general review.

Demonstration: The four triangles thus formed are equal. The triangles *1* and *2* are equal (PROP. II.); for *A H* is equal to *H B*, by hypothesis; the angle *A H E* is equal to the angle *H B F*, since they are opposite exterior and interior angles on the same side; and, for a similar reason, the angle *H A E* is equal to the angle *B H F*. (BOOK I., PROP. XVII.) The triangles *2* and *3* are equal, by PROP. I.; for *H E* and *B F* being corresponding parts of the triangles *1* and *2*, have just been proved equal; the side *H F* is common, and the angle *E H F* is equal to the angle *H F B*, its alternate angle. (BOOK I., PROP. XVI.) The triangles *3* and *4* are equal, by PROP. II.; for *E F* is common; the angle *H E F* is equal to its alternate angle *E F C*, as, also, *H F E* is equal to its alternate angle *F E C*. (BOOK I., PROP. XVI.)

The four triangles *1*, *2*, *3*, and *4* have thus been proved equal; as *1* and *4*, being respectively equal to *2* and *3*, are equal to each other, by AXIOM I. *H E*, in triangle *1*. is, then, equal to *B F* and to *F C*, its corresponding side in the triangles *2* and *4*; that is, *H E*, the given line, is equal to one-half of *B C*, the second side of the triangle. *A E* and *E C*, the parts of the third side, are equal to each other, as each is equal to *H F*, their corresponding part in triangle *2*; that is, *A C*, the third side of the triangle, is bisected at *E*.

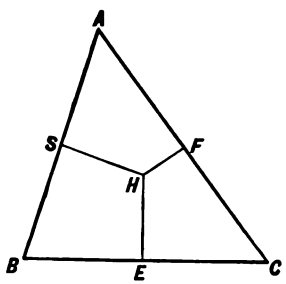
Conclusion: *A B C* being any triangle, it follows, in general, that: A straight line drawn from the middle point, etc.

Corollary I. The straight line joining the middle points of two sides of a triangle is parallel to the third side and equal to one-half of it.

Corollary II. The sides of a triangle passing through the vertices of another triangle and parallel to the opposite sides are bisected by the vertices of that triangle.

Proposition XXII.

161. Theorem: ***The perpendiculars bisecting the sides of any triangle are concurrent lines.***



Statement: Let *A B C* be any triangle, and let *E*, *F*, and *S* be the middle points of its sides. If perpendiculars are drawn to the respective sides at their middle points, *E*, *F*, and *S*, they will meet in one and the same point.

Construction: At *E* and *F*, two of the points of bisection, erect perpendiculars *E H* and *F H*. They meet, for if not, they are parallel, and then the line *F C*, perpendicular to one of them, is perpendicular to the other. (Book I., Prop. XIV.) As only one perpendicular can be drawn from a given point to a given straight line (Book I., Prop. XI.), the line *F C E* is a straight line; but this is absurd, as it is formed by two sides of the triangle. The perpendiculars then meet at some point, as *H*, Draw *H S*. The straight line *H S* is to be proved perpendicular to *A B*.

Demonstration: *H E* being perpendicular to *B C* at its middle point, *H* is equidistant from *B* and *C*. (Book I., Prop. XV.) *H F* being perpendicular to *A C* at its middle point, *H* is equidistant from *A* and *C*. (Book I., Prop. XV.) *H* is, therefore, equidistant from *A*, *B*, and *C*. By hypothesis, *S* is equidistant from *A* and *B*. As two of its points are each equally distant from *A* and *B*, the straight line *H S* is perpendicular to *A B*. (Book I., Prop. XV., Cor. III.)

The three straight lines *H E*, *H F*, and *H S* are the perpendiculars to the sides of the triangle at their middle points, and they meet in one and the same point.

Conclusion: *A B C* being any triangle, it follows, etc.

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162. Exercises. Theorems. 1. The points of bisection of the sides of an equilateral triangle may be made the vertices of a second equilateral triangle.

2. If a perpendicular from the base of an isosceles triangle intersects one of the equal sides and the other side produced, the exterior triangle thus formed is an isosceles triangle.

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163. Problems. 1. Construct an isosceles triangle, having given the base, and the sum of the base and the perpendicular from the vertical angle to the base.

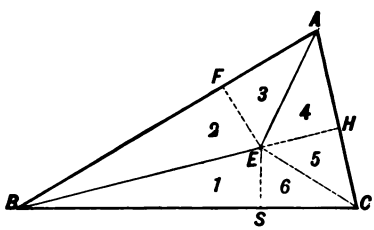
2. Construct the triangle, the perpendicular drawn from the vertical angle to the base, and the difference between each side and the adjacent segment of the base made by the perpendicular, being given.

3. Construct an isosceles triangle, the base and the sum of one of the equal sides and the perpendicular from the vertical angle to the base being given.

4. The value of an angle in equilateral triangles?

PROPOSITION XXIII.

164. Theorem: ***The straight lines bisecting the angles of any triangle are concurrent lines.***



Statement: Let *A B C* be any triangle, and let *A E* and *B E* be two straight lines bisecting the angles *A* and *B*. If a straight line is drawn from *E*, their point of intersection, to *C*, the vertex of the third angle, it will bisect that angle, and the three lines bisecting the three angles of the triangle meet in one and the same point.

Construction: From *E*, draw *E F*, *E S*, and *E H* perpendicular, respectively, to *A B*, *B C*, and *A C*.

Demonstration: The triangles *1* and *2* are equal; for, being right-angled triangles, the hypotenuse, *E B*, is common, and the angles at *B* are equal, by hypothesis. The side *F E* is, therefore, equal to *E S*. (PROP. v.)

The triangles *3* and *4* are also equal; the hypotenuse, *A E*, being common, and the angles at *A* being equal, by hypothesis. *E H* is, therefore, equal to *E F*. (PROP. v.) The lines *E H* and *E S* are, therefore, equal. (AXIOM I.)

The triangles *5* and *6* are equal; for, being right-angled triangles, the hypotenuse, *E C*, is common, and the sides *E H* and *E S* have just been proved equal. (PROP. IV.) The angles at *C* are, therefore, equal, and the line *E C* bisects the angle *C*, the third angle of the triangle; that is, the three lines bisecting the three angles meet at the point *E*.

Conclusion: *A B C* being any triangle, it follows, in general, that: The straight lines bisecting the angles, etc.

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165. Exercises. THEOREMS. 1. The sum of the sides of an isosceles triangle is less than the sum of the sides of any other triangle on the same base and between the same parallels.

2. Half of the sum of the angles at the base equals the angle made by one of the sides of the triangle, and the perpendicular from one extremity of the base to the bisectrix of the vertical angle.

3. Half of the difference of the angles at the base equals the angle made by the base and the perpendicular from one extremity of the base to the bisectrix of the vertical angle.

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166. Exercises. PROBLEMS. 1. To determine a point in a line given in position, to which lines drawn from two given points may have the greatest possible difference.

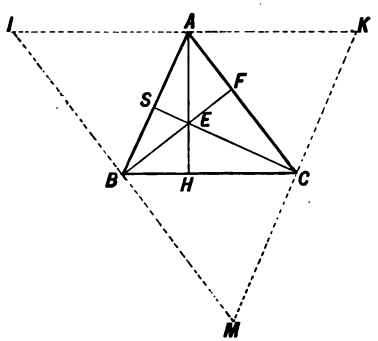
2. From two points on the same side of a line given in position, draw two lines which shall meet at a point in this line, their sum being less than the sum of any two lines drawn from the same points, and terminated at any other point in the same line.

3. An angle, a side opposite to it, and the difference of the two other sides, being given, construct the triangle.

4. An angle, a side opposite to it, and the sum of the two other sides, being given, construct the triangle.

PROPOSITION XXIV.

167. Theorem: ***The straight lines drawn from the vertices of any triangle, perpendicular to the opposite sides, are concurrent lines.***



Statement: Let *A B C* be any triangle. Let *A H*, *B F*, and *C S* be three perpendiculars from the vertices to the respectively opposite sides of the triangle. They all meet at one and the same point, as at *E*; that is, the three perpendiculars from the vertices to the opposite sides are concurrent lines.

Construction: Through *A*, *B*, and *C*, the three vertices, draw straight lines, *I K*, *I M*, and *K M*, parallel to the respectively opposite sides of the triangle *A B C*.

Demonstration: The sides of the triangle *I K M*, in passing through the vertices of the triangle *A B C*, parallel to the sides of *A B C*, opposite the respective vertices, are bisected by the vertices. (PROP. XXI., COR. II.) *A*, *B*, and *C* are, then, the middle points of *I K*, *I M*, and *K M*.

Since *A B* and *K M* are parallel, *S C*, which, by construction, is perpendicular to *A B*, is also perpendicular to *K M* at *C*, its middle point. (BOOK I., PROP. XIV.) In similar manner, *A H* is shown to be perpendicular to *I K* at *A*, its middle point; and *B F*, to be perpendicular to *M I* at *B*, its middle point.

The three straight lines *A H*, *B F*, and *C S* are, then, perpendicular, respectively, to the three sides of the triangle *I K M* at their middle points; and, therefore, the three lines *A H*, *B F*, and *C S* meet in one and the same point (PROP. XXII.); but these three lines are drawn from the vertices of the given triangle perpendicular to the opposite sides.

Conclusion: *A B C* being any triangle, it follows, etc.

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168. Exercises. THEOREMS. 1. The sum of the three lines from any point within a triangle to the three vertices is less than the sum of the three sides of the triangle, and greater than half the sum.

2. The sum of the medians is less than the sum of the sides of a triangle, and greater than half the sum.

3. In any triangle, the sides of the vertical angle being unequal, the median drawn from the vertical angle lies between the bisectrix of the vertical angle and the longer side.

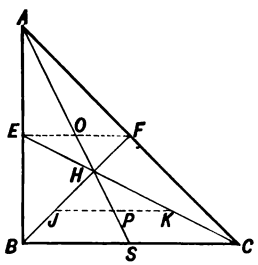
4. In any triangle, the sides of the vertical angle being unequal, its bisectrix lies between the median and the perpendicular from the vertical angle to the base.

5. The triangle made by joining points in the sides of an equilateral triangle equidistant from the vertices, in the same order, is an equilateral triangle.

6. The bisectrix of any one angle of a triangle and the bisectrices of the angles exterior to the other angles of the triangle are concurrent.

PROPOSITION XXV.

169. Theorem: ***The medians, or straight lines drawn from the vertices to the middle points of the opposite sides of a triangle, are concurrent lines.***



Statement: Let *A B C* be any triangle. The three straight lines drawn from *A*, *B*, and *C* to the middle points of the respectively opposite sides of the triangle will meet in one and the same point.

Construction: Draw *B F* from *B* to the middle point of *A C*, and *C E* from *C* to the middle point of *A B*, intersecting *B F* in *H*. Draw *A H*, and produce it to *S* in *B C*. The point *S* is to be proved the middle point of *B C*. Draw *E F*; and, also, draw *J K* from the middle point of *B H* to the middle point of *H C*.

Demonstration: *E F* is parallel to *B C* and equal to one-half of it. (PROP. XXI., COR. I.) For a similar reason, *J K* in the triangle *H B C*, is parallel to *B C* and equal to one-half of it. *E F* and *J K* are, then, equal and parallel. (BOOK I., PROP. XIX.)

In similar manner, *E O* and *J P* are proved to be equal, each being equal to one-half of *B S;* and *O F* and *P K* are equal, each being equal to one-half of *S C.* (PROP. XXI.)

The four lines *E O, O F, J P,* and *P K* may, however, be proved equal to one another. For the triangles *E H F* and *J H K* are equal (PROP. II.), since *E F* has been proved equal and parallel to *J K;* and the alternate angles at *E* and *K,* and, also, at *F* and *J,* are equal. (BOOK I., PROP. XVI.) *E H* is, therefore, equal to *H K.* The two triangles *E O H* and *H P K* are, then, proved to be equal (PROP. II.), *E H* and *H K* having just been proved equal, as also the angles *O E H* and *H K P;* and the angles *E H O* and *P H K* are equal, as they are opposite or vertical angles. (BOOK I., PROP. VII.) The side *E O* is, then, equal to *P K.*

E O has, however, been proved equal to *J P,* and *O F* to *P K.* As *E O* is equal to *P K,* the four lines are equal to one another. (AXIOM I.)

E O was proved to be equal to one-half of *B S,* and *O F* to one-half of *S C. B S* and *S C* are, therefore, equal, since their halves are equal. (AXIOM V.) *S* is, then, the middle point of *B C.* The three lines *A S, B F,* and *C E,* drawn from the vertices to the middle points of the opposite sides, meet, then, in one and the same point, and are thus shown to be concurrent lines.

Conclusion: *A B C* being any triangle, it follows, in general, that: The medians, or straight lines drawn from the vertices to the middle points of the opposite sides of a triangle, are concurrent lines.

MISCELLANEOUS EXERCISES.

THEOREMS.

170. Prove the following: I. 1. If from any point in the bisectrix of an angle, a parallel to one of the sides is drawn, the bisectrix, the parallel, and the remaining sides form an isosceles triangle.

II. 2. If through each of the vertices of a given triangle a line be drawn parallel to the opposite side, a new triangle will be formed equal to four times the given triangle.

3. The bisectrices of the vertical angle of a triangle and of one of the exterior angles at the base contain an angle equal to one-half of the other angles at the base.

III. 4. The three medians of a triangle intersect at a point of trisection, two-thirds of each lying toward the angle from which the median is drawn.

5. If through the vertices of a triangle three bisectrices of the exterior angles be drawn, the three smaller new triangles and the whole triangle thus formed will be mutually equiangular.

PROBLEMS.

Constructions: Construct triangles, the following parts being given:

I. 6. The perpendicular from any vertex to the opposite side of an equilateral triangle.

7. The base, the difference between the sides, and the difference between the angles at the base.

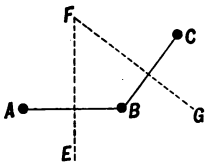
II. 8. One side and the medians to the other sides.

III. 9. The three medians.

GENERAL SCHOLIA.

INTERSECTION OF LOCI.

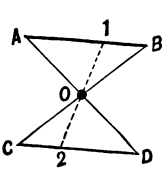
171. The method of *intersection* of *loci* is helpful in solving many determinate problems. For instance, in finding the position of a point that fulfills two conditions, while, from each condition, a line that is a *locus* of the point may be found, the point itself, if the problem is possible, will be found at the intersection of the two loci.



Thus, to find a point at equal distances from three given points, ***A***, ***B***, and ***C***. ***EF***, perpendicular to the line ***AB***, at its middle point, is the *locus* of points equidistant from ***A*** and ***B***; and ***FG***, similarly, is the *locus* of points equally distant from ***B*** and ***C***. ***F***, the *intersection of the loci*, is the required point. When is the problem impossible?

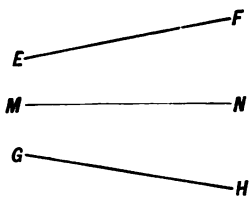
SYMMETRY.

172. The relative position of points, lines, figures, and solids with respect to some point, line, or plane of reference requires mention; thus,



Two points are diametrically opposite from any point in the straight line that joins them. If the given points are *equally distant* from the assumed point, they are diametrically *symmetrical* with respect to that point, as a *center* of *symmetry*. Every point in ***AB***, as ***1***, is diametrically symmetrical to some point, as ***2***, in ***CD***.

173. Any two points, lines, figures, or solids, or any two parts of figures or solids, are symmetrical, when for every point on one side of an intermediate point, or line, or plane, there is a diametrically or vertically opposite point at an equal distance on the other side.



Two points are vertically opposite from any intermediate line or plane perpendicular to the straight line that joins them. If the given points are *equally distant* from the assumed line or plane, they are vertically *symmetrical* with respect to the line or plane as an *axis* or *plane* of *symmetry*. Every point in **E F** is vertically symmetrical with respect to some point in **G H**.

BOOK III.—QUADRILATERALS.

SYNOPSIS.*

SECTION I.—EQUAL SIDES.

174. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. I. The opposite sides of a parallelogram are equal, each to each.

PROP. II. If the opposite sides of a quadrilateral are equal, each to each, the figure is a parallelogram.

PROP. III. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

PROP. IV. If two adjacent sides of a rectangle are equal, the figure is a square.

PROP. V. If two adjacent sides of a rhomboid are equal, the figure is a rhombus.

3. MISCELLANEOUS EXERCISES.

* The student should work Book III. without referring to the text.

SECTION II.—VALUE OF ANGLES.

175. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. VI. The four angles of a quadrilateral are together equal to four right angles.

PROP. VII. The opposite angles of a parallelogram are equal, each to each.

PROP. VIII. If the opposite angles of a quadrilateral are equal, each to each, the figure is a parallelogram.

PROP. IX. If any angle of a parallelogram is a right angle, the figure is a rectangle.

PROP. X. If any angle of a parallelogram is an oblique angle, the figure is a rhomboid.

3. MISCELLANEOUS EXERCISES.

SECTION III.—DIAGONALS AND DIAMETERS.

176. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XI. The diagonals of a parallelogram mutually bisect.

PROP. XII. The diameters of a parallelogram mutually bisect.

PROP. XIII. If the diagonals of a quadrilateral mutually bisect, the figure is a parallelogram.

PROP. XIV. The diagonals of a square are equal, and bisect each other at right angles.

PROP. XV. The diagonals of a rhombus are unequal, but bisect each other at right angles.

3. MISCELLANEOUS EXERCISES.

SECTION IV.—EQUAL AREAS.

177. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XVI. A parallelogram is divided by either of its diagonals into two equal triangles.

PROP. XVII. Parallelograms, that may be divided by their diagonals so as to make a triangle in the one equal to a triangle in the other, are equal.

PROP. XVIII. A parallelogram is divided by either of its diameters into two equal parallelograms.

PROP. XIX. A trapezoid is equivalent to any parallelogram with the same altitude and a base equal to the diameter joining the sides that are not parallel.

PROP. XX. Parallelograms with the same, or with equal bases and altitudes, are equivalent.

3. MISCELLANEOUS EXERCISES.

SECTION V.—COMPARISON OF AREAS.

178. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XXI. The square of the sum of two lines is equal to the sum of the squares of the lines, increased by twice the rectangle of the lines.

PROP. XXII. The square of the difference of two lines is equal to the sum of the squares of the lines, diminished by twice the rectangle of the lines.

PROP. XXIII. The rectangle of the sum and difference of two lines is equal to the difference of the squares of the lines.

PROP. XXIV. The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the two other sides.

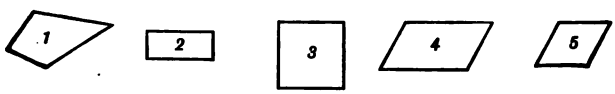
PROP. XXV. The square of any side of an oblique-angled triangle is equal to the sum of the squares of the two other sides, increased or decreased, according as the angle opposite the given side is obtuse or acute, by twice the rectangle of the one and the projection upon it of the other of the two sides.

3. MISCELLANEOUS EXERCISES.

GENERAL SCHOLIA.

179. 1. MAXIMA AND MINIMA. 2. THEORY OF LIMITS.

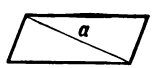
SECTION I.—EQUAL SIDES.



DEFINITIONS.

180. A **quadrilateral** is a figure bounded by **four lines.**

181. A **parallelogram** is a quadrilateral whose opposite sides are parallel ; as *2, 3, 4.*



182. A **rectangle** is a right-angled parallelogram ; **as 2.**

183. A **square** is an equilateral rectangle ; as *3.*

184. A **rhomboid** is an oblique-angled parallelogram ; as *4.*

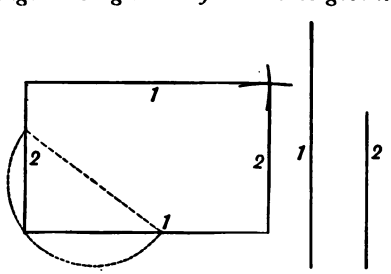
185. A **rhombus** is an equilateral rhomboid ; as *5.*

A *diagonal* is a straight line joining vertices not adjacent ; as *a.*

RULE AND COMPASS.

186. 1. *Construct a rectangle having two adjacent sides given.*

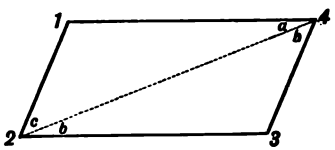
Make *1* equal to one of the sides, and at one of its extremities erect a perpendicular, extending it so that it shall equal *2*, the other side. From the outer extremity of *2* draw an arc at a distance equal to *1*, and from the outer extremity of *1*, an arc at a distance equal to *2*. With the rule, complete the quadrilateral.



2. *Construct a square on a given line.*

PROPOSITION I.

187. Theorem: ***The opposite sides of a parallelogram are equal, each to each.***



Statement: Let *1 2 3 4* be any parallelogram. The opposite sides *1 4* and *2 3* are equal to each other; and, also, the opposite sides *1 2* and *3 4* are equal to each other.

Construction: Draw the diagonal *2 4*.

Demonstration: The two triangles *1 2 4* and *2 4 3* have, by construction, the side *2 4* common; the alternate angles *a* and *b*, equal; and, also, the alternate angles *c* and *h*, equal. (BOOK II., PROP. XVI.) The two triangles, having two angles and the included side of the one equal to two angles and the included side of the other, each to each, are equal. (BOOK II., PROP. II.) The sides *1 4* and *2 3*, opposite the equal angles, *c* and *h*, are, then, equal; and the sides *1 2* and *4 3*, opposite the equal angles, *a* and *b*, are equal.

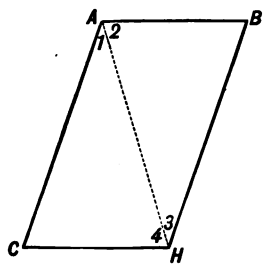
Conclusion: *1 2 3 4* being any parallelogram, it follows, etc.

Corollary I. Parallel lines included between parallel lines are equal.

Corollary II. Parallel lines are everywhere equally distant.

PROPOSITION II.

188. Theorem: ***If the opposite sides of a quadrilateral are equal, each to each, the figure is a parallelogram.***



Statement: Let *A B H C* have the opposite sides *A B* and *C H* equal; and, also, the opposite sides *A C* and *B H* equal. Prove that the quadrilateral is a parallelogram.

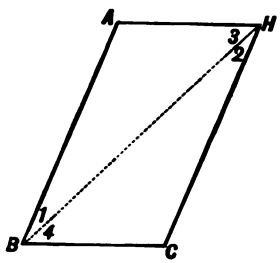
Construction: Draw the diagonal *A H*.

Demonstration: In the two triangles *A C H* and *A B H*, the side *A H* is common, by construction; the sides *A C* and *B H* are equal, by hypothesis; as are also the sides *C H* and *A B*. The two triangles are, therefore, equal. (BOOK II., PROP. III.) The alternate angles, *1* and *3*, opposite the equal sides, *C H* and *A B*, are thus proved to be equal; and, therefore, the sides *A C* and *B H* are parallel. (BOOK II., PROP. XVIII.) The alternate angles, *2* and *4*, opposite the equal sides, *B H* and *A C*, are equal, and, therefore, the sides *A B* and *C H* are parallel. The opposite sides of the quadrilateral being parallel, each to each, it is, by definition, a parallelogram.

Conclusion: *A B H C* being any quadrilateral, it follows, etc.

PROPOSITION III.

189. Theorem: ***If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.***



Statement: Let *A B C H* be any quadrilateral, having two opposite sides, as *A B* and *H C*, equal and parallel. The figure is to be proved a parallelogram.

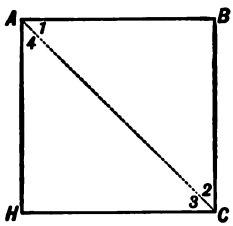
Construction: Draw the diagonal *B H*.

Demonstration: In the triangles *A B H* and *B H C*, the side *B H* is common, by construction; the sides *A B* and *H C* are equal, by hypothesis; and the alternate angles *1* and *2* are equal, the sides *A B* and *H C* being parallel. (BOOK I., PROP. XXIV.) The two triangles are, therefore, equal. (BOOK II., PROP. I.) The angles *3* and *4*, opposite the equal sides *A B* and *H C*, are thus proved equal. The angles *3* and *4* being equal, the sides *A H* and *B C* are parallel. (BOOK I., PROP. XVIII.) The figure *A B C H*, having its opposite sides parallel, each to each, is, by definition, a parallelogram.

Conclusion: *A B* and *H C* being any two equal, etc.

PROPOSITION IV.

190. Theorem: *If two adjacent sides of a rectangle are equal, the figure is a square.*



Statement: Let *A B C H* be a rectangle with two of its adjacent sides, as *A B* and *B C*, equal. The four sides are then equal to one another, and the rectangle is a square.

Construction: Draw the diagonal *A C*.

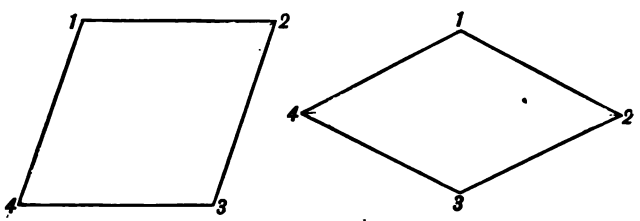
Demonstration: In the two triangles *A B C* and *A H C*, the side *A C* is common; the angles *1* and *3* are equal, as they are alternate angles, the sides *A B* and *C H* being parallel, by hypothesis (BOOK I., PROP. XVI.); and, similarly, the angles *4* and *2* are equal, *A H* and *B C* being parallel.

The two triangles, having two angles and the included side of the one equal to two angles and the included side of the other, each to each, are equal. (BOOK II., PROP. II.) The sides *H C* and *A H* are thus proved equal, respectively, to the sides *A B* and *B C*. The sides *A B* and *B C* are, however, equal to each other. The sides *A B*, *B C*, *A H*, and *H C* are, then, equal to one another. (AXIOM I.)

Conclusion: *A B C H* being any rectangle, it follows, etc.

PROPOSITION V.

191. Theorem: ***If two adjacent sides of a rhomboid are equal, the figure is a rhombus.***



Statement: Let *1 2 3 4* be any rhomboid, and let two of its adjacent sides, as *1 4* and *1 2*, be equal to each other. The four sides are to be proved equal to one another, and the figure is thus shown to be a rhombus.

Demonstration: A rhomboid being, by definition, a parallelogram, the figure *1 2 3 4* is a parallelogram. The opposite sides are, therefore, equal. (PROP. I.) The side *4 3* is, then, equal to the side *1 2*, and *2 3* is equal to *1 4*. Since, by hypothesis, the sides *1 2* and *1 4* are equal to each other, the four sides are equal to one another. (AXIOM I.) The figure is, then, by definition, a rhombus.

Conclusion: As *1 2 3 4* is any rhomboid, it follows, etc.

Exercise. In an isosceles triangle, $A = \frac{1}{2} B$, or $\frac{1}{2} C$; find A, B, and C. $A = 4 B$, or $4 C$; find A, B, and C.

MISCELLANEOUS EXERCISES.

192. 1. Straight lines drawn through any point in a diagonal of a parallelogram, parallel to its sides, divide the parallelogram into four parallelograms, two of which must be equal.

The two equal parallelograms to which reference is made are sometimes called *complements*. If the given point bisects the given diagonal, the *four* parallelograms are equal to one another.

2. A diagonal of any parallelogram is trisected by two straight lines intersecting it, drawn from one of the opposite angles to the middle points of the two opposite sides.

3. If a straight line be drawn through any vertex of a parallelogram so as to intersect the parallelogram, the perpendicular to it from the opposite vertex is equal to the difference between the perpendiculars to it from the other vertices.

4. The straight line joining the points of bisection of two sides of a triangle is parallel to the third side.

5. If a straight line be drawn through any vertex of a parallelogram so as not to intersect the parallelogram, the perpendicular to the line from the opposite vertex is equal to the sum of the perpendiculars to it from the other vertices.

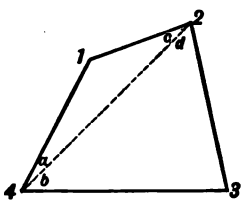
6. The four straight lines that join the points of bisection of the adjacent sides of any quadrilateral form a parallelogram.

7. Any straight line drawn from the vertex to the base of a triangle, or to the base produced, is bisected by the straight line (produced, if necessary) that joins the middle points of the two other sides of the triangle.

SECTION II.—ANGLES.

PROPOSITION VI.

193. Theorem: ***The four angles of a quadrilateral are together equal to four right angles.***



Statement: Let *1 2 3 4* be any quadrilateral. Prove the sum of its angles, *1, 2, 3,* and *4,* equal to four right angles.

Construction: Draw the diagonal *2 4.*

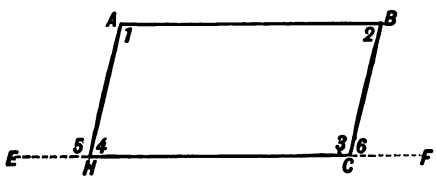
Demonstration: By construction, the quadrilateral has been divided into two triangles. It is obvious, that the four angles of the quadrilateral are together equal to the six angles of the two triangles. The three angles of each triangle are, however, equal to two right angles. (BOOK II., PROP. XVII.) The six angles of the two triangles, or the four angles of the quadrilateral, are, then, equal to four right angles. (AXIOM V.)

Conclusion: *1 2 3 4* being any quadrilateral, etc.

Exercise. 1. Given three angles of any quadrilateral, find the fourth angle. Find the sum of the exterior angles.

PROPOSITION VII.

194. Theorem: ***The opposite angles of a parallelogram are equal, each to each.***



Statement: Let *A B C H* be any parallelogram. Prove that the opposite angles *1* and *3* are equal to each other; and, also, that the opposite angles *2* and *4* are equal to each other.

Construction: Extend the base *H C* to *E* and to *F*, making the exterior angles *5* and *6*.

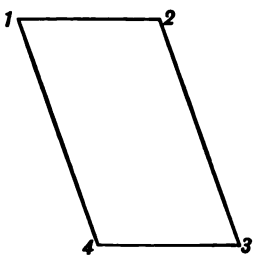
Demonstration: The angle *1* is equal to the angle *5*, as they are alternate angles. (BOOK II., PROP. XVI.) The angle *3* is equal to the angle *5*, as they are opposite exterior and interior angles on the same side. (BOOK II., PROP. XVII.) The opposite angles *1* and *3*, each being equal to the angle *5*, are equal to each other. (AXIOM I.)

In similar manner, it may be proved that the opposite angles *2* and *4* are each equal to the angle *6*, and, therefore, they are equal to each other.

Conclusion: As *A B C H* is any parallelogram, it follows, in general, that: The opposite angles of a parallelogram are equal, each to each.

PROPOSITION VIII.

195. Theorem: ***If the opposite angles of a quadrilateral are equal, each to each, the figure is a parallelogram.***



Statement: Let *1 2 3 4* be any quadrilateral, with the angle *1* equal to its diagonally opposite angle *3*, and the angle *2* equal to the angle *4*. Prove that *1 2 3 4* is a parallelogram.

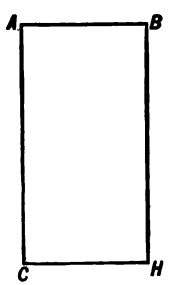
Demonstration: Since the angles *4* and *2* are equal, it is obvious, that the angle *4* is half of their sum. Since the angles *3* and *1* are equal, the angle *3* is half of their sum. The angles *4* and *3* are, then, together equal to half of the sum of the four angles of the quadrilateral, or two right angles. (PROP. VI.) In similar manner, any other two consecutive angles, as *1* and *4*, may be proved equal to two right angles.

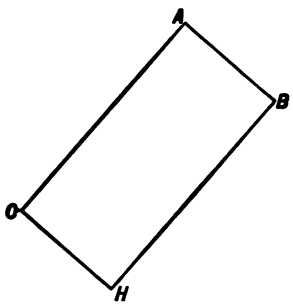
Since the angles *1* and *2* are together equal to two right angles, the sides *1 4* and *2 3* are parallel. (BOOK I., PROP. XVIII., COR. II.) Since *1* and *4* are together equal to two right angles, *1 2* and *4 3* are parallel. The opposite sides being parallel, the figure is, by definition, a parallelogram.

Conclusion: The figure *1 2 3 4* being any quadrilateral, **etc.**

PROPOSITION IX.

196. Theorem: *If any angle of a parallelogram is a right angle, the figure is a rectangle.*





Statement: Let *A B H C* be any parallelogram. Let any angle, as *A*, be a right angle. Prove the figure a rectangle.

Demonstration: The angles *A* and *B* are together equal to two right angles. (BOOK I., PROP. XVII.) As *A* is a right angle, *B* is, also, a right angle. (AXIOM IV.)

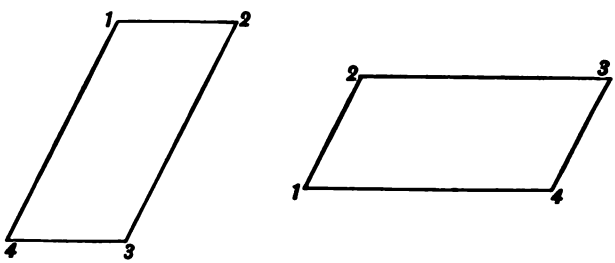
As *A* and *B* are right angles, their respectively opposite angles, *H* and *C*, are right angles. (PROP. VII.) The four angles are, then, all right angles, and the figure is a rectangle.

Conclusion: The figure *A B H C* being any parallelogram, and the angle *A* any one of its angles, it is true, in general, that: If one angle of a parallelogram, etc.

Exercise. 1. If one side of a quadrilateral be extended in both directions, the sum of the exterior angles is equal to the sum of the two opposite interior angles.

PROPOSITION X.

197. Theorem: ***If any angle of a parallelogram is an oblique angle, the figure is a rhomboid.***



Statement: Let *1 2 3 4* be any parallelogram in which any angle, as *1*, is an oblique angle. The other angles, *2*, *3*, and *4*, are then also oblique, and the parallelogram is a rhomboid.

Demonstration: Since the figure *1 2 3 4* is a parallelogram, the sides *1 4* and *2 3* being parallel and intersected by the side *1 2*, the angles *1* and *2* are together equal to two right angles. (BOOK I., PROP. XVII.) The angle *1*, being oblique, is either greater or less than a right angle ; hence, the angle *2* is less or greater than a right angle. (AXIOM XII.)

The angles *1* and *2* being oblique, their respectively opposite angles, *3* and *4*, are also oblique, since they are equal to them. (PROP. VII.)

Conclusion: *1 2 3 4* being any parallelogram, it follows, etc.

Exercise. 1. The angles included between the sides of a quadrilateral produced equal four right angles.

MISCELLANEOUS EXERCISES.

198. 1. The bisectrices of the interior angles of a parallelogram form a rectangle.

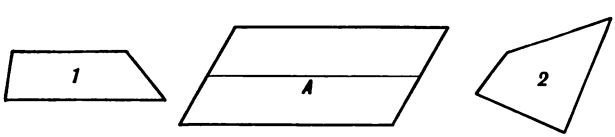
2. The bisectrices of the interior angles of a rectangle form a square.

3. If the points in the sides of a square, at equal distances from the vertices taken in their order, be joined by straight lines, these straight lines will themselves form a square.

4. If the angles adjacent to one base of a trapezoid are equal, those adjacent to the other base are also equal.

5. If from any point in the base of an isosceles triangle parallels to the equal sides are drawn, the perimeter of the parallelogram thus formed is equal to the sum of the equal sides of the triangle.

SECTION III.—DIAGONALS AND DIAMETERS.



DEFINITIONS.

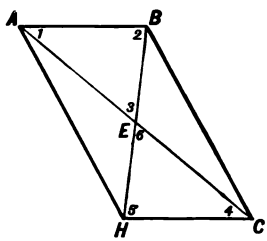
199. A **diameter** of a quadrilateral is a straight line joining the middle points of two opposite sides, as *A*.

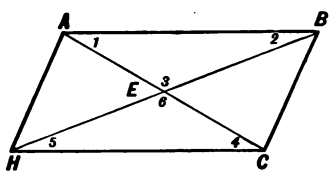
200. A **trapezoid** is a quadrilateral with only two parallel sides, as *1*.

201. A **trapezium** is a quadrilateral with none of its sides parallel, as *2*.

PROPOSITION XI.

202. Theorem: ***The diagonals of a parallelogram mutually bisect.***





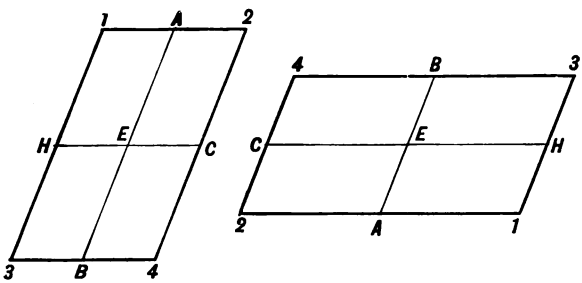
Statement: Let *A B C H* be any parallelogram, with its diagonals, *A C* and *B H*, intersecting at the point *E*. The diagonal *A C* is divided by the diagonal *B H* into two equal parts, *A E* and *E C*; and the diagonal *B H* is divided by the diagonal *A C* into two equal parts, *B E* and *E H*.

Demonstration: The two triangles *A E B* and *H E C* have the sides *A B* and *H C* equal (PROP. I.); and, the angles *2* and *5* equal to each other, and, also, the angles *1* and *4*. (BOOK I., PROP. XVI.) The triangles *A E B* and *H E C*, having two angles and the included side of the one equal to two angles and the included side of the other, each to each, are equal. (BOOK II., PROP. II.) The sides *A E* and *E C*, opposite the equal angles *2* and *5*, are, therefore, equal to each other; and, also, the sides *B E* and *E H*, opposite the equal angles *1* and *4*, are equal to each other. The diagonals *A C* and *B H* are thus shown to bisect each other at *E*, the point of intersection.

Conclusion: *A C* and *B H* being diagonals of *A B C H*, any, etc.

PROPOSITION XII.

203. Theorem: ***The diameters of a parallelogram mutually bisect.***



Statement: Let *1 2 4 3* be any parallelogram, with its diameters, *A B* and *H C*, intersecting at *E*. Prove that *A E* and *E B* are equal; and, also, *H E* and *E C*.

Demonstration: *1 2* and *3 4* being opposite parallel sides, by hypothesis, are equal. (PROP. I.) *1 A*, which is half of *1 2*, is, then, equal to *3 B*, which is half of *3 4*. (AXIOM VI.) As *1 A* and *3 B* are, therefore, equal and parallel, *1 3* and *A B*, joining their extremities, are also equal and parallel. (PROP. III.) As *A B* is parallel to *1 3*, it is also parallel to *2 4*. (BOOK I., PROP. XIV.)

In similar manner, *H C* is proved equal and parallel to *1 2* and *3 4*.

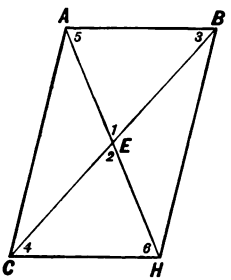
The diameter *H C* is divided at *E* into two equal parts; for *H E* and *E C*, its two parts, being respectively equal to *1 A* and *A 2* (PROP. I.), they are equal to each other. (AXIOM I.)

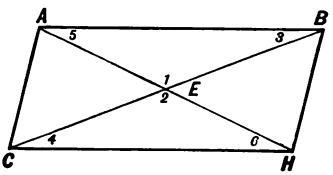
In similar manner, it is proved that *A B* is bisected at *E*.

Conclusion: *A B* and *H C* being diameters of *1 2 4 3*, any, etc.

PROPOSITION XIII.

204. Theorem: ***If the diagonals of a quadrilateral mutually bisect, the figure is a parallelogram.***





Statement: Let *A B H C* be a quadrilateral, whose diagonals, *A H* and *B C*, bisect each other at *E*. *A B H C* is a parallelogram.

Demonstration: The triangles *A E B* and *C E H* have the sides *A E* and *E H* equal, by hypothesis; and, also, *B E* and *E C*. And the vertical angles *1* and *2* are equal. (BOOK I., PROP. VII.) The triangles *A E B* and *C E H* are, therefore, equal. (BOOK II., PROP. I.) The angles *3* and *4*, being opposite the equal sides *A E* and *E H*, are thus proved equal; as, also, *5* and *6*, opposite the equal sides *B E* and *E C*; and the side *A B* is thus proved equal to *C H*.

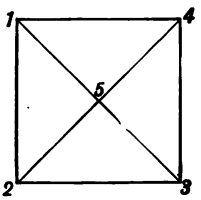
Since the angles *3* and *4* are equal, *A B* and *C H* are parallel. (BOOK I., PROP. XVIII.) *A B* and *C H* having thus been shown to be equal and parallel, the figure *A B C H* is a parallelogram. (PROP. III.)

Conclusion: *A H* and *B C* being the diagonals, etc.

NOTE.—If diameters mutually bisect, the figure is not necessarily a parallelogram.

PROPOSITION XIV.

205. Theorem: ***The diagonals of a square are equal, and bisect each other at right angles.***



Statement: Let *1 2 3 4* be any square. The diagonals *1 3* and *2 4* are equal, and bisect each other at right angles.

Demonstration: The triangles *1 2 3* and *2 3 4* are equal, having two sides and the included angles, respectively, equal. (BOOK II., PROP. I.) Therefore, *1 3* and *2 4* are equal.

The diagonals *1 3* and *2 4*, moreover, bisect each other, since *1 2 3 4* is a parallelogram. (PROP. XI.)

The four triangles, having *5* as a common vertex, are, then, all equal. (BOOK II., PROP. III.) The angles at *5* are all equal, and each is a right angle. (BOOK I., PROP. VI., COR. II.)

Conclusion: *1 2 3 4* being any square, it follows, etc.

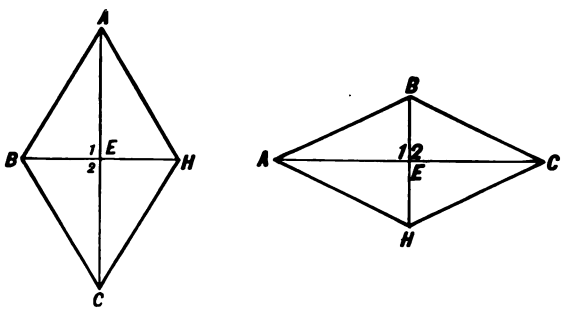
Corollary I. The diagonals of a rectangle are equal, but bisect each other obliquely.

Exercises. 1. If the diagonals of a quadrilateral are equal, and bisect each other at right angles, the figure is a square.

2. If the diagonals of a quadrilateral are equal, but bisect each other obliquely, the figure is a rectangle.

PROPOSITION XV.

206. Theorem: ***The diagonals of a rhombus are unequal, but bisect each other at right angles.***



Statement: Let *A B C H* be any rhombus. Its diagonals, *A C* and *B H*, are unequal, but bisect each other at right angles.

Demonstration: Since the figure is a rhombus, its angles *B* and *C* are oblique; and as, together, they are equal to two right angles (BOOK I., PROP. XVII.), one is obtuse and the other acute. The triangles *A B C* and *B C H* have two sides of the one equal to two sides of the other, each to each, but one included angle, *B*, greater than the other, *C*. *A C* is, then, greater than *B H*. (BOOK II., PROP. XIV.) The diagonals are unequal; and, as the figure is a parallelogram, the diagonals bisect each other. (PROP. XI.)

The triangles *A B E* and *B E C* are equal. (BOOK II., PROP. III.) The angles *1* and *2* are, then, right angles. (BOOK I., PROP. VI.) The diagonals are perpendicular to each other.

Conclusion: *A B C H* being any quadrilateral, it follows, etc.

MISCELLANEOUS EXERCISES.

207. 1. If the diagonals of a quadrilateral are unequal, but bisect each other at right angles, the figure is a rhombus.

2. If the diagonals of a quadrilateral are unequal, and bisect each other obliquely, the figure is a rhomboid.

3. The sum of the diagonals of a trapezium is less than the sum of any four lines which can be drawn to the vertices from any point within the figure, except the intersection of the diagonals.

4. The sum of the angles of a four-sided figure is twice as great as the sum of the angles of a three-sided figure; the sum of the angles of a five-sided figure, three times as great; of a six-sided figure, four times as great; and so on.

5. The lines which join the middle points of the sides of a quadrilateral, successively, form a parallelogram equal to one-half of the quadrilateral, and whose perimeter is equal to the sum of the diagonals of the quadrilateral.

6. If, through the extremities of each diagonal of a quadrilateral, parallels to the other diagonal be drawn, a parallelogram double the given quadrilateral will be formed.

7. The perimeter of a quadrilateral is greater than the sum of its diagonals.

8. If a diagonal of a quadrilateral bisects two of its angles, it is perpendicular to the other diagonal.

9. If the diagonals of a parallelogram are equal, the figure is a rectangle.

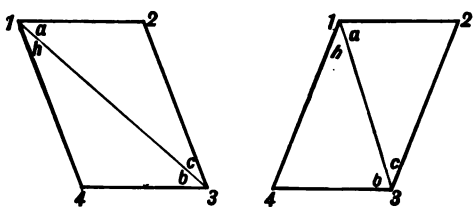
10. If the diagonals of a quadrilateral bisect its angles, the figure is a rhombus.

11. Between two intersecting straight lines, draw a straight line of a given length, and parallel to a given straight line.

SECTION IV.—EQUAL AREAS.

PROPOSITION XVI.

208. Theorem: *A parallelogram is divided by either of its diagonals into two equal triangles.*



Statement: Let *1 2 3 4* be any parallelogram, with either of its diagonals, as *1 3*, given. The triangles *1 2 3* and *1 4 3*, into which the parallelogram is thus divided, are equal, and, therefore, equivalent.

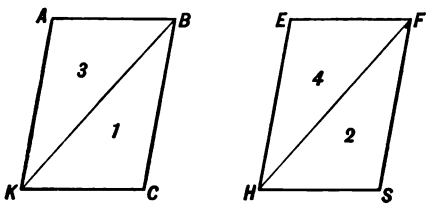
Demonstration: The triangles *1 2 3* and *1 4 3* have *1 3* in common; and the alternate angles *a* and *b* equal, as, also, the alternate angles *h* and *c*. (BOOK I., PROP. XVI.) The triangles are, therefore, equal. (BOOK II., PROP. II.)

In proving triangles equal, it is shown that they may be made to coincide throughout, and they are, therefore, equal in area, or equivalent.

Conclusion: *1 2 3 4* being any parallelogram, and *1 3* being either of its diagonals, it follows, in general, that: A parallelogram is divided by either of its diagonals into two equal triangles.

PROPOSITION XVII.

209. Theorem: ***Parallelograms, that may be divided by their diagonals so as to make a triangle in the one equal to a triangle in the other, are equal.***



Statement: Let *A B C K* and *E F S H* be two parallelograms, divided by their diagonals, *B K* and *F H*, so as to have the triangle *B C K*. in the one, equal to the triangle *F S H*, in the other. The parallelograms are to be proved equal.

Demonstration: The triangles *B C K* and *F S H* may be equal, by any one of the first five propositions of Book II.

As the triangles *1* and *2* are equal, the triangles *3* and *4* are, also, equal, being equal, respectively, to *1* and *2* (AXIOM I.); since a diagonal divides a parallelogram into two equal triangles. (PROP. XVI.)

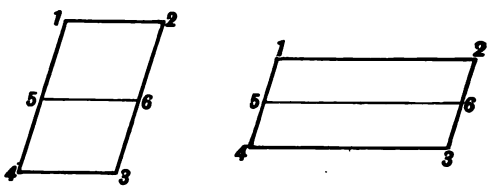
The four triangles being equal, it is obvious that the parallelograms may be made to coincide, and are equal.

Conclusion: *A B C K* and *E F S H* being, etc.

Corollary I. Parallelograms having two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, are equal.

PROPOSITION XVIII.

210. Theorem: *A parallelogram is divided by either of its diameters into two equal parallelograms.*



Statement: Let *1 2 3 4* be any parallelogram, with either of its diameters, as *5 6*, given. The two parts *1 2 6 5* and *5 6 3 4* are to be proved equal parallelograms.

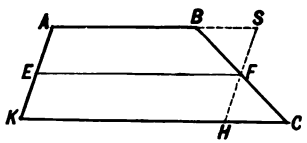
Demonstration: The sides *1 4* and *2 3*, being opposite sides, are equal. (PROP. I.) Their halves, *1 5* and *2 6*, are, then, equal; and *5 6* and *1 2*, joining corresponding extremities of lines equal and parallel, are themselves equal and parallel. (PROP. III.) Similarly, *5 6* is equal and parallel to *4 3*.

The figures *1 2 6 5* and *5 6 3 4* are, then, parallelograms. They have *1 2* and *1 5*, of the one, respectively equal to *5 6* and *5 4*, of the other. The included angles, *5 1 2* and *4 5 6*, are also equal, being opposite exterior and interior angles on the same side. The parallelograms *1 2 6 5* and *5 6 3 4*, into which the given parallelogram is divided, are, then, equal. (PROP. XVII., COR. I.)

Conclusion: *1 2 3 4* being any parallelogram, etc.

PROPOSITION XIX.

211. Theorem: *A trapezoid is equivalent to any parallelogram with the same altitude, and a base equal to the diameter joining the sides that are not parallel.*



Statement: Let *A B C K* be any trapezoid, the sides *A B* and *K C* being parallel, and let *E F* be the diameter joining the other sides. Prove the trapezoid equivalent to any parallelogram with the same altitude, and base equal to *E F*.

Construction: Through *F*, draw *S H* parallel to *A K*, and extend *A B* to *S*.

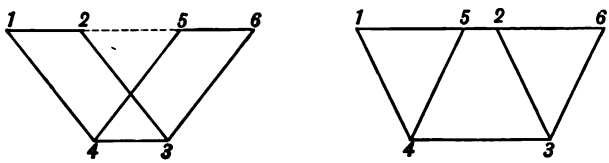
Demonstration: The triangles *B F S* and *H F C* have the angles at *F* equal (BOOK I., PROP. VII.); the side *B F* equal to *F C*, by hypothesis; and the sides *S F* and *F H* equal (AXIOM I.), being equal, respectively, to *A E* and *E K*. (PROP. VII.) The triangles *B F S* and *H F C* are, therefore, equal. (BOOK II., PROP. I.)

The entire figure *A S F C K* = *A S F C K*; and it has just been proved that *B S F* = *H F C*. By equal subtractions (AXIOM IV.), *A B C K* = *A S H K*; that is, the trapezoid is equivalent to the parallelogram having the same altitude and a base, *K H*, equal to the diameter *E F*. (PROP. I.)

Conclusion: *A B C K* being any trapezoid, it follows, etc.

PROPOSITION XX.

212. Theorem: ***Parallelograms with the same, or with equal bases and altitudes, are equivalent.***



Statement: Let *1 2 3 4* and *5 6 3 4* be any two parallelograms, with the same base, or equal bases, *4 3*, and equal altitudes. The parallelograms are equivalent.

Construction: If the parallelograms have different bases, place the one upon the other, so that their bases coincide in position.

Demonstration: Since the parallelograms have the same altitude, the upper bases are at the same distance from the lower base; and, being parallel to it and on the same side of it, they form one and the same straight line. (BOOK I., PROP. XIX., COR. I.)

The triangles *1 4 5* and *6 3 2* are equal (BOOK II., PROP. I.); for *1 4* is equal to *2 3*, and *4 5* is equal to *3 6*, being opposite sides of parallelograms (PROP. I.); and the angles at *4* and *3* are equal. (BOOK I., PROP. XX.)

From the entire figure *1 4 3 6*, subtracting the equal triangles, leaves the given parallelograms, *1 2 3 4* and *5 6 3 4*, equal. (AXIOM IV.)

Conclusion: As *1 2 3 4* and *5 6 3 4* are any, etc.

Corollary I. A triangle is equivalent to half of any parallelogram with equal base and altitude.

Corollary II. Triangles with the same, or with equal bases and altitudes, are equivalent.

Corollary III. With given base, the greater altitude makes the greater parallelogram or triangle.

MISCELLANEOUS EXERCISES.

213. 1. Bisect a parallelogram by a line drawn from a point in one of its sides.

2. The perimeter of an isosceles triangle is greater than the perimeter of a rectangle which is of the same altitude with, and equivalent to, the given triangle.

3. If the middle points in the sides of any quadrilateral figure are joined by four straight lines, the figure so formed will be a parallelogram.

4. If, from any point within a parallelogram, lines be drawn to the four vertices, either set of opposite triangles thus formed is equivalent to half the parallelogram.

5. If two triangles have two sides of the one equal to two sides of the other, each to each, and if the included angles are supplementary, the two triangles will be equivalent.

6. Two quadrilaterals are equal when they have one diagonal and four sides, or two diagonals and three sides equal, each to each.

7. The four bisectrices of the angles of a quadrilateral, not a square, form a second quadrilateral, whose opposite angles are supplementary.

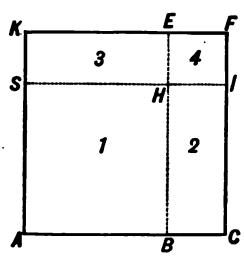
SECTION V.—COMPARISON OF AREAS.

DEFINITION.

214. A **projection** of a line is that part of a second line included between perpendiculars to it, from the extremities of the first line. Illustrate.

PROPOSITION XXI.

215. Theorem: ***The square of the sum of two lines is equal to the sum of the squares of the lines, increased by twice the rectangle of the lines.***



Statement: Let *AB* and *BC* be any two straight lines, and let *AC* be their sum. Let *ACFK* be the square of *AC*. The square *ACFK* is equal to the sum of the squares of *AB* and *BC*, increased by twice the rectangle of *AB* and *BC*.

Construction: Divide *AK* at *S* into parts, *AS* and *SK*, respectively equal to *AB* and *BC*. Draw *SI*, a common parallel to *AC* and *KF*; and *BE*, a common parallel to *AK* and *CF*.

Demonstration: The figures *1*, *2*, *3*, and *4* are parallelograms, by construction; and, the angles *A*, *C*, *F*, and *K* being right angles, the figures are rectangles. (PROP. IX.)

The rectangle *1* is, however, the square of *A B*, for the sides *A S* and *A B* are equal, by construction. (PROP. IV.)

The rectangle *2* is the rectangle of *B H* and *B C*; but, since the figure *1* is a square, *B H* is equal to *A B*. The figure *2* is, therefore, the rectangle of *A B* and *B C*. (AXIOM I.)

The rectangle *3* is the rectangle of *K S* and *K E*; or, of *A B*, a parallel and equal projection of *K E* (PROP. I.), and *B C*, by construction equal to *K S*.

The rectangle *4* is a square (PROP. IV.); for the sides *F E* and *F I* are equal, being equal to their respectively parallel and equal projections, *B C* and *K S*. (PROP. I) The rectangle *4* is, then, the square of *F E*; or, of *B C*. (AXIOM I.)

Since the whole is equal to the sum of all its parts, *A C F K* is equal to the sum of the areas *1*, *2*, *3*, and *4*; that is, the square of *A C* is equal to the square of *A B* plus twice the rectangle of *A B* and *B C* plus the square of *B C*.

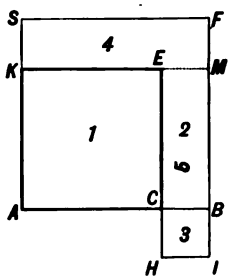
Conclusion: *A B* and *B C* being any two lines, **etc.**

Corollary I. The square of a line is equal to the sum of the rectangles, with the given line as one side, and the several parts of the given line as the other side.

Corollary II. The rectangle of any two lines equals the sum of the rectangles, with one of the lines as one side, and the parts of the other line as the other side.

PROPOSITION XXII.

216. Theorem: ***The square of the difference of two lines is equal to the sum of the squares of the lines, decreased by twice the rectangle of the lines.***



Statement: Let *A C* be the difference between the two lines *A B* and *B C*. Let *A E* be the square of *A C*; *A F*, the square of *A B*; and *B H*, the square of *B C*. Prove that *A E* equals the sum of *A F* and *B H*, diminished by twice the rectangle of *A B* and *B C*.

Construction: Extend the line *K E* to *M*, a point in *B F*.

Demonstration: The lines *A K* and *A S* form one and the same straight line (BOOK I., PROP. XI.); and *F B I* and *E C H* are straight lines. (BOOK I., PROP. VIII.)

The square *1* is evidently equal to the entire figure, or the sum of the squares of *A B* and *B C*, diminished by the figures *4* and *5*.

The figure *4* is a parallelogram (PROP. III.), as *S F* and *K M*, being each parallel and equal to *A B* (PROP. I.), are parallel and equal to each other. (BOOK I., PROP. XIX.) The angle *S*,

moreover, being a right angle, the figure is, also, a rectangle. (PROP. IX.) It is the rectangle of ***S F*** and ***S K***; but ***S F*** is equal to ***A B*** (PROP. I.); and ***S K*** is equal to ***C B*** (AXIOM IV.). The figure *4* is, then, equal to the rectangle of ***A B*** and ***B C***.

Similarly, the figure *5* is proved to be a parallelogram and a rectangle. It is the rectangle of ***E H*** and ***H I***; but ***H I*** is equal to ***B C*** (PROP. I.); and ***E H*** is equal to ***E C*** plus ***C H***. ***E C***, one of the parts of ***E H***, is equal to ***A C***; and ***C H***, the other part, is equal to ***B C***, being sides of the same square. ***E H*** is, therefore, equal to ***A B***. The figure *5* is, therefore, equal to the rectangle of ***A B*** and ***B C***.

The square of the difference of two lines is thus proved equal to the entire figure, or the sum of the squares of the lines, diminished by twice the rectangle of the lines.

Conclusion: ***A B*** and ***B C*** being any lines, it follows, etc.

Exercises. 1. If an equilateral triangle be constructed on one side of a given line, and on the other side two equilateral triangles, one on each half of the line, then the two lines drawn from the vertex of the larger triangle to the vertices of the smaller triangles will trisect the given line.

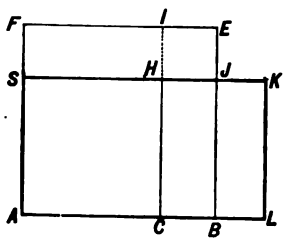
2. Divide a given triangle into three equivalent parts by lines drawn from the vertices of the triangle to a point within it.

3. If a straight line be divided into any two parts, the rectangle of the whole line and one of the parts is equal to the square of that part, together with the rectangle of the two parts.

4. Parallelograms about the diagonals of a square are themselves squares.

PROPOSITION XXIII.

217. Theorem: ***The rectangle of the sum and difference of two lines is equal to the difference of the squares of the lines.***



Statement: Let *A B* and *B C* be any two lines; and let *A C* be the difference, and *A L* the sum, of the lines. Let *A K* be the rectangle of *A L* and *A C*; *A H*, the square of *A C*; and *A E*, the square of *A B*. Prove *A K* equal to the difference between *A E* and *A H*.

Construction: Extend the line *C H* to *I*.

Demonstration: The lines *A S* and *A F* form one and the same straight line; also, *S H* and *S K*. (BOOK I., PROP. XI.)

The figures *F H*, *I J*, *H B*, and *J L*, are all parallelograms, as, the lines *I C*, *E B*, and *K L* being each parallel to *A F*, they are parallel to one another; and, the lines *F E* and *S K* being each parallel to *A L*, they are parallel to each other. (BOOK I., PROP. XIX.)

The figures are, also, rectangles; as they have each, by construction, a right angle. (PROP. IX.) The figure *I J*, moreover, has, by equal subtractions, the side *H J* equal to

the side HI (AXIOM IV.); IJ is, therefore, a square. (PROP. IV.) It is the square on HJ, or its equal, CB, the difference of the two lines.

The rectangles FH and HB are equal (PROP. XVIII.); but the rectangle HB is equivalent to JL. (PROP. XVII.) FH is, therefore, equivalent to JL. (AXIOM I.)

	$AK = AJ + JL$	(AXIOM II.)
but,	$JL = FH$	
therefore,	$AK = AJ + FH$	(AXIOM I.)
and,	$AJ + FH = AE - EH$	(AXIOM IV.)
or,	$AK = AE - EH$	(AXIOM I.)

AK is the rectangle of the sum and difference of the two lines; AE is the square of AB, one of the lines; and EH is the square of the other line, BC. The rectangle of the sum and difference of the two lines is, therefore, equal to the difference of their squares.

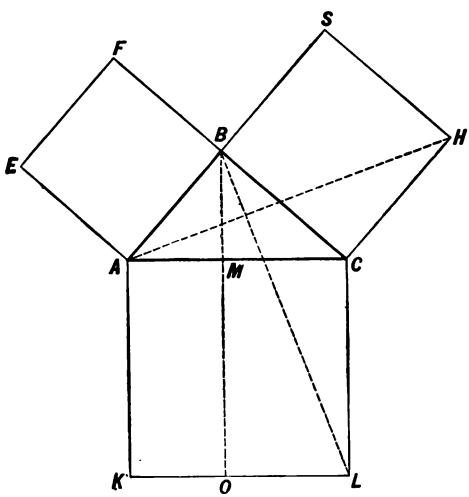
Conclusion: AB and BC being any two straight lines, it follows, in general, that: The rectangle of the sum and difference of two lines is equal to the difference of the squares of the lines.

Exercises. 1. If a straight line be divided equally and, also, unequally, the rectangle contained by the unequal parts, and the square on the line between the points of section, are together equal to the square of half the line.

2. If a straight line be divided into any two parts, four times the rectangle of the whole line and one of the parts, together with the square of the other part, is equal to the square of the straight line which is made up of the whole line and that part.

PROPOSITION XXIV.

218. Theorem: ***The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the two other sides.***



Statement: Let *ABC* be any right-angled triangle. Let *AL* be a square described on *AC*, the hypotenuse, and *BE* and *BH* squares described on *AB* and *BC*, the two other sides of the triangle. The square *AL* is equal to the sum of the squares *BE* and *BH*.

Construction: Draw *AH* and *BL*, and draw *BO* perpendicular to *KL*.

Demonstration: The lines *AS* and *CF* are straight lines (BOOK I., PROP. VIII.), as three of the angles at *B* are, by

construction, right angles; and the straight line *B O* is parallel to *A K* and to *C L*, as they are all perpendicular to *K L*. (Book I., Prop. XIII.) The parallelograms *C O* and *A O* are rectangles, each having one right angle. (Prop. IX.)

The rectangle *C O* is double the triangle *B C L*, as they have the same base, *C L*, and the same altitude, the perpendicular distance between the parallels *C L* and *B O*. (Prop. XX., Cor. I.)

The square *B H* is double the triangle *A C H*, as they have the same base, *C H*, and the same altitude, the perpendicular distance between the parallels *A S* and *C H*. (Prop. XX., Cor. I.)

The triangles *B C L* and *A C H* are, however, equal (Book II., Prop. I.), as *B C*, one side of the one triangle, is equal to *C H*, one side of the other triangle, each being a side of the square *B H*; and *C L*, a side of the one triangle, is equal to *A C*, a side of the other triangle, each being a side of the square *A L*; and the included angle *B C L*, of the one triangle, is equal to the included angle *H C A*, of the other triangle, as each is equal to a right angle plus the angle *B C A*.

Since the triangles *B C L* and *A C H* are equal, their doubles, the rectangle *C O* and the square *B H*, are equal. (Axiom V.)

In similar manner, the rectangle *A O* may be proved equal to the square *B E*. The sum of *A Q* and *C O*, or *A L*, the square of the hypotenuse, is, then, by equal additions, proved equal to *B E* plus *B H*, the sum of the squares of the two sides of the triangle. (Axiom III.)

Conclusion: *A B C* being any right-angled triangle, it follows, in general, that: The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the two other sides.

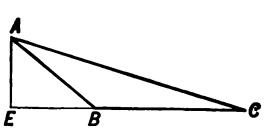
Corollary I. The square of either side of a right-angled triangle is equal to the square of the hypotenuse diminished by the square of the other side.

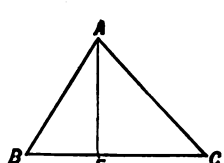
Corollary II. The square of either side of a right-angled triangle is equal to the rectangle of the hypotenuse and the projection of the given side upon the hypotenuse.

Corollary III. If the square of one side of a triangle is equal to the sum of the squares of the two other sides, the triangle is a right-angled triangle.

PROPOSITION XXV.

219. Theorem: ***The square of any side of an oblique-angled triangle is equal to the sum of the squares of the two other sides, increased or decreased, according as the angle opposite the given side is obtuse or acute, by twice the rectangle of the one and the projection upon it of the other of the two sides.***





Statement: Let *A B C* be any oblique-angled triangle. Let *A C* be any side, and let *B E* be the projection of *A B*, one of the other sides, upon *B C*, the third side. The square of *A C* is equal to the sum of the squares of *A B* and *B C*, increased by twice the rectangle *B C . B E*, when the angle *B* is obtuse; or decreased by twice the rectangle *B C . B E*, when *B* is acute.

Demonstration: I. When the opposite angle is obtuse:

$$AC^2 = AE^2 + EC^2 \qquad \text{(PROP. XXIV.)}$$

but, $AE^2 = AB^2 - EB^2$ (PROP. XXIV., COR. I.);

and, $EC^2 = BC^2 + EB^2 + 2BC \cdot EB$. (PROP. XXI.)

By substitution, $AC^2 = AB^2 - EB^2 + BC^2 + EB^2 + 2BC \cdot EB$;

by combining, $AC^2 = AB^2 + BC^2 + 2BC \cdot EB$. (*Q. E. D.*)

II. When the angle opposite the given side is acute:

$$AC^2 = AE^2 + EC^2 \qquad \text{(PROP. XXIV.)}$$

but, $AE^2 = AB^2 - BE^2$ (PROP. XXIV., COR. I.);

and, $EC^2 = BC^2 + BE^2 - 2BC \cdot BE$. (PROP. XXII.)

By substitution, $AC^2 = AB^2 - BE^2 + BC^2 + BE^2 - 2BC \cdot BE$;

by combining, $AC^2 = AB^2 + BC^2 - 2BC \cdot BE$. (*Q. E. D.*)

Conclusion: Since ABC is any oblique-angled triangle, etc.

Corollary I. The sum of the squares of any two sides of a triangle is equal to twice the square of half the third side, increased by twice the square of the median to that side.

Corollary II. The difference of the squares of any two sides of a triangle is equal to twice the rectangle of the third side and the projection of the median upon that side.

Corollary III. The sum of the squares of the sides of any parallelogram equals the sum of the squares of the diagonals.

Corollary IV. The sum of the squares of the sides of any quadrilateral is equal to the sum of the squares of the diagonals, increased by four times the square of the line joining their middle points.

MISCELLANEOUS EXERCISES.

PROBLEMS.

220. Constructions. Construct required figures, parts being given as follows :

I. 1. Square, one side.

2. Rhombus, one side and one angle.

3. Rectangle, two sides.

4. Rhomboid, two sides and included angle.

5. Rhomboid, two sides, and an angle not included.

6. Trapezoid, two adjacent sides, included angle, and angle at other extremity of given parallel base.

7. Isosceles trapezoid, two adjacent sides, and included angle.

8. Trapezium, three consecutive sides and two included angles.

9. Trapezium, two adjacent sides and three angles.

10. Triangle, base, its median, and the angle included between them.

II. 11. Square, the sum of its sides, or perimeter.

12. Square, a diagonal.

13. Rhombus, two diagonals.

14. Rhombus, perimeter and one angle.

15. Rhombus, perimeter and one diagonal.

16. Rectangle, perimeter and one side.

17. Rectangle, one side and angles included between given side and diagonal.

18. Rectangle, one diagonal and angle included between the two diagonals.

19. Rectangle, diagonal and angles on the same side at its extremities.

20. Rhomboid, perimeter, one side, and one angle.

21. Rhomboid, one side and angles included between given side and diagonals.

22. Rhomboid, two diagonals and included angle.

23. Isosceles trapezoid, bases and altitude.

24. Isosceles trapezoid, bases and one angle.

25. Isosceles trapezoid, bases and diagonal.

26. Triangle, perimeter and three angles.

27. Triangle, perimeter and angles at base.

28. Isosceles triangle, perimeter and angle at vertex.

29. Isosceles triangle, perimeter and base.

30. Isosceles triangle, perimeter and altitude.

31. Parallelogram, one diagonal and the angles on the same side of it, at its extremities, made with it by sides of the parallelogram.

32. Equilateral triangle, altitude.

33. Equilateral triangle, perimeter.

III. 34. Rectangle, shorter side and difference of two sides.

35. Rectangle, longer side and difference of two sides.

36. Rectangle, sum and difference of two sides.

37. Rhombus, shorter diagonal and difference of two diagonals.

38. Rhombus, longer diagonal and difference between two diagonals.

39. Rhombus, sum and difference of diagonals.

40. Rhombus, one side and one diagonal.

41. Triangle, the base, its median, and the difference of the angles at the base.

42. Parallelogram, one diagonal and the difference of the angles on same side of it, at its extremities, made with it by sides of the parallelogram.

43. Parallelogram, sum and difference of diagonals and the angle included between them.

44. Isosceles trapezoid, altitude and sum and difference of the parallel sides.

45. Trapezoid, sum and difference of parallel sides and sum and difference of the angles at the base.

46. Trapezoid, two adjacent sides, the included angle, and the difference between the two parallel sides.

47. Trapezoid, the altitude, the shorter parallel side, the difference between the parallel sides, and the difference between the angles at the base.

48. Trapezoid, the altitude, the longer parallel side, the difference between the parallel sides, and the difference between the angles at the base.

49. Triangle, base, sum of other sides, and difference of angles at base.

50. Triangle, three medians.

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221. Constructions. Find equivalent areas, as follows:

I. 51. Square, equal to the sum of two squares.

52. Square, equal to the difference of two squares.

53. Square, equal to one fourth the sum of two squares.

54. Square, equal to one fourth the difference of two squares.

55. Rectangle, equal to the sum of any number of rectangles of the same altitude.

56. Rectangle, equal to the difference of two rectangles of the same altitude.

57. Triangle, equivalent to a rectangle.

58. Triangle with given angle, equivalent to a rectangle.

59. Triangle, equivalent to any parallelogram.

60. Triangle with required angle, equivalent to any parallelogram.

II. 61. Triangle, equivalent to a figure of any number of sides.

62. Triangle with required angle, equivalent to any polygon, or figure of any number of sides.

63. Isosceles triangle, equivalent to any polygon.

64. Rectangle, equivalent to any polygon.

65. Parallelogram with required angle, equivalent to any polygon.

66. Parallelogram with required diagonal, equivalent to any polygon.

67. Triangle with required median to the base, equivalent to any trapezoid, or trapezium.

68. Two rectangles, respectively equivalent to two squares.

69. Rectangle on a given line, equivalent to a square.

70. Parallelogram with required angle, equivalent to a square.

71. Parallelogram with required diagonal, equivalent to a square.

III. 72. Square, equivalent to three times a given square.

73. Square, equivalent to the sum of any number of different squares.

74. Triangle with given median to the base, equivalent to any square.

75. Triangle with given median to the base, equivalent to any polygon.

76. Bisect a triangle by a straight line from any vertex to the opposite side.

77. Bisect a triangle by a straight line drawn from a given point in one of its sides.

78. Trisect any triangle by two straight lines from any vertex to the opposite side.

79. Trisect any triangle by two straight lines from any point in any side.

80. Bisect any parallelogram by a straight line drawn through any point.

81. Bisect any parallelogram by a straight line drawn perpendicular to one of the sides.

82. Bisect any parallelogram by a straight line parallel to a given line.

83. Bisect any quadrilateral by a straight line from any one of the vertices.

84. Bisect any quadrilateral by a straight line from any point in any of the sides.

THEOREMS.

222. Prove the following: 85. If, from any point in the diagonal (or diagonal produced) of a parallelogram, straight lines be drawn to the opposite angles, they will cut off equivalent triangles.

86. The area of a trapezoid is equal to half that of a parallelogram whose base is the sum of the parallel sides, and whose altitude is the perpendicular distance between those sides.

87. The two triangles formed by drawing straight lines from any point within a parallelogram to the extremities of two opposite sides are together half of the parallelogram.

88. If a straight line be drawn from one of the acute angles of a right-angled triangle, bisecting the opposite side, the square upon that line is less than the square upon the hypotenuse by three times the square of half the bisected side.

89. If, at any point of a straight line, a perpendicular be drawn to this line, the difference of the squares of the distances of any point of this perpendicular from the two extremities of the given line is constant.

90. Any straight line passing through the point of bisection of either diagonal of a parallelogram bisects the parallelogram.

91. If, from one of the acute angles of a right-angled triangle, a line be drawn to the opposite side, the squares of that side and the line so drawn are together equivalent to the squares of the hypotenuse and the segment adjacent to the right angle.

92. In any triangle, if a line be drawn from the vertex at right angles to the base, the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.

93. The squares of the diagonals of a parallelogram are together equivalent to the squares of the four sides.

94. The squares of the four sides of a parallelogram are together equivalent to the squares of the two diagonals.

95. If lines be drawn to the four vertices of a rectangle, from any point within or without the rectangle, the squares of the lines drawn to opposite vertices will together be equivalent to the squares of the other two.

96. In any triangle, if a line be drawn from the vertex bisecting the base, the sum of the squares of the two sides of the triangle is double the sum of the square of the bisecting line and the square of half the base.

97. The area of any two parallelograms described on the two sides of a triangle is equal to that of a parallelogram on the base, whose side is equal and parallel to the line drawn from the vertex of the triangle to the intersection of the two sides of the other parallelograms produced to meet.

98. The squares of the diagonals of a trapezoid are together equivalent to the squares of the non-parallel sides, and twice the rectangle contained by the parallel sides.

99. If, from any point whatever, lines be drawn to the four vertices of a parallelogram, twice the sum of their squares will be equivalent to the squares of the diagonals, together with eight times the square of the line drawn from the given point to the intersection of the diagonals.

100. In any quadrilateral, the sum of the squares of the four sides is equal to the sum of the squares of the diagonals, together with four times the square of the line joining the middle points of the diagonals.

101. Prove that, if the squares of the sides of a quadrilateral are together equivalent to the squares of the diagonals, the figure must be a parallelogram.

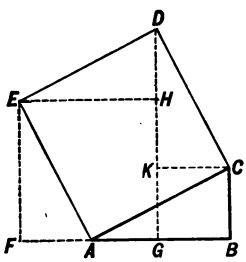
102. Four times the sum of the squares of the medians of a triangle is equal to three times the sum of the squares of the sides of the triangle.

103. If perpendiculars to any line be drawn from the three vertices of any triangle, together they will be three times the perpendicular from the intersection of the medians of the triangle to the same line.

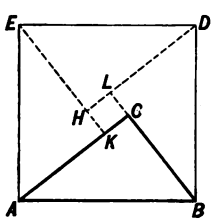
104. In any trapezium, if two opposite sides be bisected, the sum of the squares of the other two sides, together with the squares of the diagonals, is equal to the sum of the squares of the bisected sides, together with four times the square of the line joining those points of bisection.

105. The area of the triangle contained by the medians of a triangle is three-fourths of the area of the given triangle.

106. Prove Prop. XXIV., using the following diagram:



107. Prove Prop. XXIV., using the following diagram:

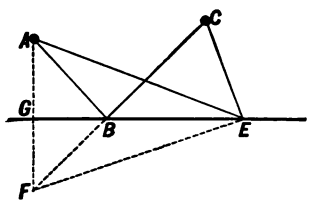


108. Prove Prop. XXIV. by any fourth method.

GENERAL SCHOLIA.

MAXIMA AND MINIMA.

223. Exercises. 1. The shortest lines drawn from two given points to any point in a given straight line are those that make equal angles with that line.



Statement: AB and CB are together shorter than AE and CE, if the angles ABG and CBE are equal.

Construction: Let AG be perpendicular to GE, the given line, and produce CB to F, in AG produced, and join FE.

Demonstration:

$\angle GBF = \angle CBE$ (Book I., Prop. VII.)

$\triangle ABG = \triangle GBF$ (Book II., Prop. II.)

$\therefore AB = BF$

and, $CF = AB + CB$

also, $GF = AG$

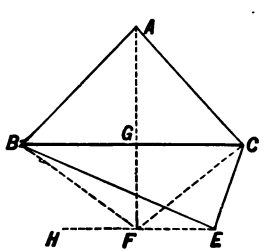
$\therefore \triangle AGE = \triangle EGF$ (Book II., Prop. I.)

and, $AE = EF$

but, $CF < CE + EF$ (Book II., Prop. VII.)

$\therefore AB + CB < CE + AE$ *Q. E. D.*

2. An isosceles triangle is the maximum triangle that can be constructed on a given base, the sum of the other sides of the triangle being constant.



Statement: Let BC be a given base. Let AB equal AC, and their sum equal the sum of BE and EC, any two unequal lines. The triangle BAC is greater than the triangle BEC.

Construction: Place the triangles as in the diagram. Draw AG perpendicular to BC. Draw EF perpendicular to AG produced, and draw BF and FC.

Demonstration: As $AB = AC$, the $\triangle AGB = \triangle AGC$. (BOOK II.) PROP. IV.) AF, then, is perpendicular to BC at its middle point, and $BF = FC$. (BOOK I., PROP. XV.) $\angle BFG = \angle CFG$ (BOOK I., PROP. XXIV., COR. II.); and, $\therefore \angle BFH = \angle CFE$; but,

$$BE + EC > BF + FC \qquad \text{(EXERCISE I.)}$$
$$\therefore AB + AC > BF + FC \qquad \text{(AXIOM I.)}$$
$$\therefore AB > BF \qquad \text{(AXIOM VI.)}$$
$$\therefore AG > GF \qquad \text{(BOOK I., PROP. XXV.)}$$
$$\therefore \triangle BAC > \triangle BFC \qquad \text{(PROP. XX., COR. III.)}$$
but, $$\triangle BFC = \triangle BEC \qquad \text{(PROP. XX., COR. II.)}$$
$$\therefore \triangle BAC > \triangle BEC. \qquad Q.\ E.\ D.$$

3. The maximum triangle with two sides given is that in which the given sides include a right angle.*

THEORY OF LIMITS.

224. Imagine that a straight line can be changed according to some simple law of change; thus, suppose that one extremity remains fixed in position, while the other is moved in either direction indicated by the straight line. The length of the line is then a *variable* quantity, only one of its points being *constant*.

The movable extremity may evidently be moved in the one direction until it coincides with the fixed extremity, and in the other direction it may be moved toward infinity. There are, thus, two limits: (*a*) the fixed point; (*b*) the line produced indefinitely toward infinity.

225. When the variable is increasing, it approaches its *superior limit*; when it decreases, it approaches its *inferior limit*.

* For other exercises on Maxima and Minima, see pp. 160 and 193.

Any limit, whether point, line, angle, figure, or solid, with finite extension from the point of observation or of comparison, may be called a *finite limit;* with infinite extension, it is an *infinite limit.*

226. An infinite limit can never be reached ; it is, therefore, always, only an *approximate limit.*

227. A finite limit may or may not be reached, according to the conditions of the problem, and the particular supposition that may be made ; that is, a finite limit may be *absolute,* or *approximate.*

1. The law of change may allow one of the extremities of the line freedom of motion in either of the assigned directions. It is obvious, then, that the one extremity may be made to coincide with the other extremity. The line thus becomes a point. The point is the *absolute limit* of the line.
2. The law of change may allow an intermediate point in a finite line to move in each unit of time one-half of the distance between it and one of the extremities. It is obvious, from the statement of the problem, that the moving point cannot reach the extremity. The limit is an *approximate limit.* It is the same with numbers ; $.33\frac{1}{3}$ has, for an approximate limit, $\frac{1}{3}$.

228. The **theory of limits** is, then, that lines, angles, figures, and solids, in changing their parts according to certain laws of change, reach, or approximate, under one or more particular suppositions, certain limits prescribed by the conditions of the given problem.

229. It has been seen that the motion of the parts of a figure not only gives limits but may give symmetry.

The question of maxima and minima, *greatest* and *smallest,* figures or forms may be raised in connection with either limits or symmetry. Limits will always give maxima or minima. In symmetrical figures or forms, maxima or minima may also be expected.

BOOK IV.—CIRCLES.

SYNOPSIS.

SECTION I.—EQUAL CHORDS AND ARCS.

230. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. I. Every diameter of a circle divides the circle and its circumference into two equal parts.

PROP. II. In equal circles, equal arcs are subtended by equal chords.

PROP. III. In equal circles, equal chords subtend equal arcs.

PROP. IV. In equal circles, equal chords are at equal distances from the centers ; and, conversely, chords at equal distances from the centers are equal.

PROP. V. In any circle, parallel chords intercept equal arcs.

3. MISCELLANEOUS EXERCISES.

SECTION II.—UNEQUAL CHORDS AND ARCS.

231. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. VI. A diameter of a circle is greater than any other chord.

PROP. VII. In equal circles, the greater chord subtends the greater arc.

PROP. VIII. In equal circles, the greater arc is subtended by the greater chord.

PROP. IX The greater of any two chords in a circle is at the less distance from the center.

PROP. X. Of two chords in a circle, the chord at the less distance from the center is the greater chord.

3. MISCELLANEOUS EXERCISES.

SECTION III.—CHORDS AND PERPENDICULARS.

232. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XI. A radius perpendicular to a chord of a circle bisects that chord and also the arc subtended by the chord.

PROP. XII. A radius that bisects a chord, or that bisects the arc of a chord, is perpendicular to that chord.

PROP. XIII. A straight line perpendicular to a radius at its outer extremity is tangent to the circle at that point ; and, conversely, a tangent is perpendicular to the radius drawn to the point of contact.

PROP. XIV. A straight line perpendicular to a tangent of a circle at the point of contact, or perpendicular to a chord at its middle point, passes through the center of the circle.

PROP. XV. Through any three points not in the same straight line, one circumference of a circle, and but one, can be described.

3. MISCELLANEOUS EXERCISES.

SECTION IV.—ANGLES AND ARCS.

233. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XVI. In equal circles, equal angles at the center are subtended by equal arcs ; and, conversely, equal arcs subtend equal angles at the center.

PROP. XVII. If an angle at the center of a circle and an angle at the circumference are subtended by the same arc, the angle at the circumference is equal to half of the angle at the center.

PROP. XVIII. All the angles inscribed in semicircles are right angles.

PROP. XIX. All the angles inscribed in a given segment of a circle, greater than a semicircle, are equal acute angles.

PROP. XX. All the angles inscribed in any segment of a circle, less than a semicircle, are equal obtuse angles.

3. MISCELLANEOUS EXERCISES.

SECTION V.—TWO CIRCLES.

234. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XXI. If the distance between the centers of two circles is equal to the sum of their radii, the circles are tangent externally.

PROP. XXII. If the distance between the centers of two circles is equal to the difference of their radii, the circles are tangent internally.

PROP. XXIII. If the circumferences of two circles intersect, the distance between the centers is less than the sum and greater than the difference of the radii.

PROP. XXIV. A circle may be inscribed in, or circumscribed about, any regular polygon.

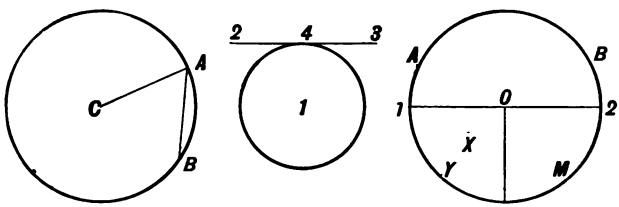
PROP. XXV. If a regular polygon of any number of sides be inscribed in a circle, a regular polygon of the same number of sides may be circumscribed about the circle; or, if a regular polygon of any number of sides be circumscribed about a circle, a regular polygon of the same number of sides may be inscribed in the circle.

3. MISCELLANEOUS EXERCISES

GENERAL SCHOLIA.

235. 1. CONCYCLIC POINTS. 2. INSCRIBED AND ESCRIBED CIRCLES. 3. THE TRIANGLE AND NINE OF ITS CIRCLES. 4. THE NINE-POINTS CIRCLE.

SECTION I.—EQUAL CHORDS AND ARCS.



DEFINITIONS.

236. A **circle** is a plane figure bounded by a curved line, all the points of which are equally distant from a point within, called the *center;* as the circle whose center is *C*.

237. The **circumference** is the line that bounds the circle.

238. A **radius** is any straight line extending from the center to the circumference of a circle ; as *C A*.

239. A **diameter** of a circle is any straight line passing through the center and terminated in both directions by the circumference ; as *1 2* in the circle *A B M*.

240. A **semicircle** is half of a circle, and a **semi-circumference** is half of a circumference.

241. A **quadrant** is a quarter of a circle, as *X;* or a quarter of a circumference, as *Y*.

242. An **arc** is any portion of a circumference ; as *A B*, or *A 2 B*, in the circle whose center is *0*.

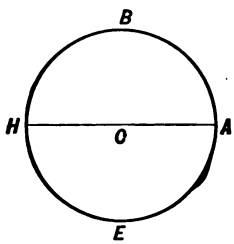
243. A **chord** is a straight line joining the extremities of any arc ; as the chord *A B*.

A chord *subtends* (*stretches under*) its arc.

244. A line is a **tangent** to a circle when it meets the circumference, but, being produced, does not cut it ; as *2 3*.

PROPOSITION I.

245. Theorem: ***Every diameter of a circle divides the circle and its circumference into two equal parts.***



Statement: Let *H B A E* be a circle. Let *C* be the center of the circle, and *H C A* one of the diameters.

The diameter divides the circle into two equal areas and the circumference into two equal arcs.

Construction: With *H A* as an axis, turn the part *H B A* upon the part *H E A*.

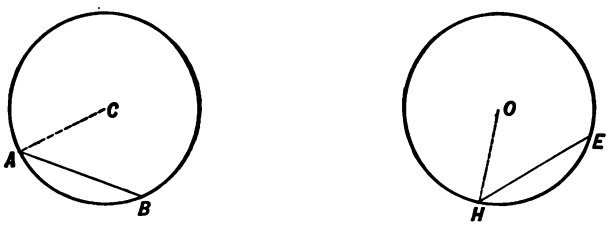
Demonstration: The two arcs *H B A* and *H E A* will coincide throughout their entire extent; for, by the construction, they coincide at their extremities. They coincide, also, between these points; for, if they do not, they will have points unequally distant from the center, which is contrary to the definition of a circumference. (DEF.) As the areas and the arcs coincide, they are equal. (AXIOM I.)

Conclusion: *H B A E* being any circle, and *H A* being, etc.

Corollary I. Diameters divide circles into semicircles, and circumferences into semi-circumferences.

PROPOSITION II.

246. Theorem: ***In equal circles, equal arcs are subtended by equal chords.***



Statement: The circles whose centers are *C* and *O* are equal; also, the arcs *AB* and *HE*. Prove the chords *AB* and *HE* equal.

Construction: Draw the radii *CA* and *OH*. Place the circle whose center is *O* upon the circle whose center is *C*, so that *O* shall fall upon *C*, and *OH* take the direction of *CA*.

Demonstration: As the circles are equal, the radii are equal. The point *H*, therefore, falls upon the point *A*. The two circumferences, moreover, coincide throughout; otherwise, they have points unequally distant from the center.

The arc *HE* falls, therefore, upon the arc *AB*, and, since they are equal, the point *E* falls on the point *B*.

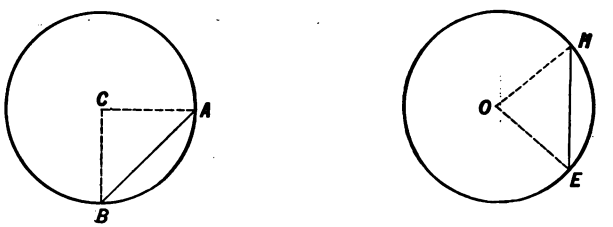
As *H* falls upon *A*, and *E* upon *B*, the chord *HE* coincides with the chord *AB*. (POST. 1.)

Conclusion: The circles whose centers are *C* and *O*, etc.

Corollary I. In the same circle, equal arcs are subtended by equal chords.

PROPOSITION III.

247. Theorem: ***In equal circles, equal chords subtend equal arcs.***



Statement: The circles whose centers are *C* and *O* are equal; also, the chords *AB* and *HE*. Prove the arcs *AB* and *HE* equal.

Construction: Draw the radii *CA*, *CB*, *OH*, and *OE*.

Demonstration: In the triangles *CAB* and *OHE*, the sides *CA*, *CB*, *OH*, and *OE* are equal, being equal radii; and the third sides, *AB* and *HE*, are equal, by hypothesis. The angles *C* and *O* are, therefore, equal. (BOOK II., PROP. III.)

Place the one circle upon the other, so that the angle *C* shall coincide with *O*. The points *A*, *B*, and *C* then coincide with *H*, *E*, and *O*. (BOOK I., PROP. IV.) The arcs *AB* and *HE*, therefore, coincide and are equal; otherwise, there would be points unequally distant from the center.

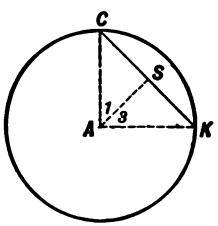
Conclusion: The circles whose centers are *C* and *O*, etc.

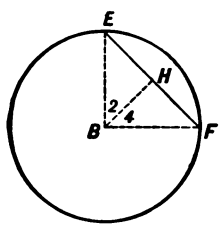
Corollary I. In the same circle, equal chords subtend equal arcs.

Corollary II. Equal circles coincide, if the centers are made to coincide.

Proposition IV.

248. Theorem: ***In equal circles, equal chords are at equal distances from the centers; and, conversely, chords at equal distances from the centers are equal.***





Statement: Let the circles whose centers are *A* and *B* be equal. If the chords *C K* and *E F* are equal, they are equally distant from the centers; and, conversely, if they are equally distant from the centers, they are equal.

Construction: Draw *A S* and *B H* perpendicular, respectively, to *C K* and to *E F*. Draw the radii *A C*, *A K*, *B E*, and *B F*.

Demonstration: I. Since the chords are equal, the triangles *A C K* and *B E F* are equal, and may be made to coincide. (Book II., Prop. III.) The perpendiculars, *A S* and *B H*, also, coincide (Book I., Prop. XII.); that is, the distances of the chords from the centers are equal.

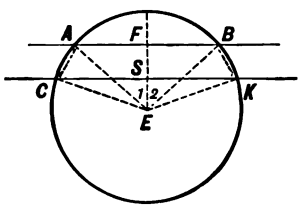
II. If the perpendiculars are equal, the four right-angled triangles are equal (Book II., Prop. IV.), and *E H* plus *H F*, or *E F*, equals *C S* plus *S K*, or *C K*; that is, the chords are equal.

Conclusion: The circles whose centers are *A* and *B*, etc.

Corollary I. In the same circle, equal chords, etc.

PROPOSITION V.

249. Theorem: *In any circle, parallel chords intercept equal arcs.*



Statement: Let the chords, or secants, *AB* and *CK* be parallel. The intercepted arcs, *AC* and *BK*, are equal.

Construction: Draw the chords *AC* and *BK*; also, the radii *EA*, *EB*, *EC*, *EK*, and the line *EF* perpendicular to the chord *AB*.

Demonstration: *EF*, being perpendicular to *AB*, is perpendicular to *CK*. (BOOK I., PROP. XIII.) The radii *AE* and *BE* being equal, the angles *1* and *2* are equal. (BOOK I., PROP. XXIV., COR. II.) For a similar reason, the angles *CES* and *KES* are equal. By equal subtractions, the angle *CES* less the angle *AEF*, or the angle *AEC*, is equal to the angle *KES* less the angle *BEF*, or the angle *BEK*. (AXIOM IV.) The triangles *AEC* and *BEK* are, then, equal in all their parts. (BOOK II., PROP. I.) The chords *AC* and *BK* are, therefore, equal. Since the chords *AC* and *BK* are equal, the arcs *AC* and *BK* are equal. (PROP. III., COR. I.)

Conclusion: The circle whose center is *E* being any circle it follows, in general, that: In any circle, parallel chords intercept equal arcs.

MISCELLANEOUS EXERCISES.

250. 1. Chords that join corresponding extremities of two parallel chords are equal.

2. Intersecting chords joining the extremities of two parallel chords are equal.

3. Chords joining corresponding extremities of equal arcs or chords are parallel.

4. Intersecting chords that join the extremities of equal arcs or chords are equal.

5. If two chords which intersect each other have one part equal in each, the two chords are also equal.

6. If, on each side of any point in the circumference of a circle, any number of equal arcs be taken, and the extremities of each pair be joined, the sum of the chords so drawn will be equal to the last chord produced to meet a line drawn from the given point through the extremity of the first arc.

7. The perpendiculars from the vertices of a triangle to the opposite sides are the bisectrices of the angles of the triangle formed by joining the feet of the perpendiculars.

8. Find the locus of the middle points of all chords of a given length that can be drawn in a given circle.

9. The tangents drawn through the vertices of an inscribed rectangle form a rhombus.

10. Through a given point without a circle, draw a secant with equal internal and external segments.

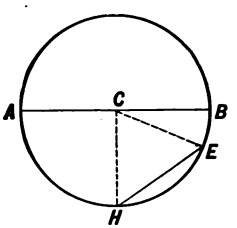
11. Find the locus of the middle points of the secants drawn from a given point to a given circumference.

12. All chords of a circle touching an interior concentric circle are equal, and are bisected at the point of contact.

SECTION II.—UNEQUAL CHORDS AND ARCS.

PROPOSITION VI.

251. Theorem: ***A diameter of a circle is greater than any other chord.***



Statement: Any diameter, as ***A B***, of the circle whose center is ***C***, is greater than ***H E***, any other chord in that circle.

Construction: Draw the radii ***C H*** and ***C E*** to the extremities of the chord ***H E***.

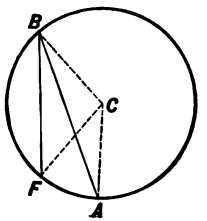
Demonstration: In the triangle ***H C E***, the sides ***C H*** and ***C E*** together are greater than the side ***H E***. (BOOK II., PROP. XI.) The sides ***C H*** and ***C E*** are, however, radii, and are, therefore, together equal to the diameter ***A B***. (DEF.) By substitution, then, ***A B*** is greater than ***H E***. (AXIOM I.)

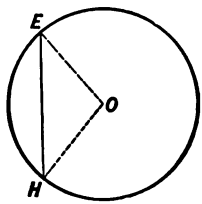
Conclusion: The circle whose center is ***C*** being any circle, it follows, in general, that: A diameter of a circle, etc.

Exercise. A straight line can intersect the circumference of a circle in only two points.

PROPOSITION VII.

252. Theorem: *In equal circles, the greater chord subtends the greater arc.*





Statement: Let the circles whose centers are *C* and *O* be equal, and let the chord *BA* be greater than the chord *EH*. Prove the arc *BA* greater than the arc *EH*.

Construction: Draw the radii *CB*, *CA*, *OE*, and *OH*.

Demonstration: In the triangles *CBA* and *OEH*, the sides *CB* and *CA* are equal to the sides *OE* and *OH*; and, by hypothesis, *BA* is greater than *EH*. The angle at *C* is, then, greater than the angle at *O*. (BOOK II., PROP. XV.)

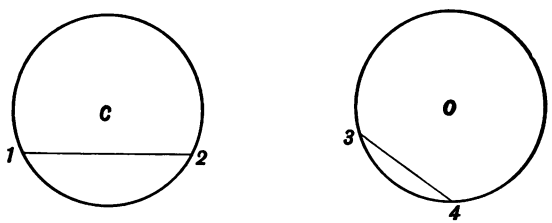
Let the circles be made to coincide so that the points *O* and *E* shall fall upon the points *C* and *B*. Since the angle at *O* is less than the angle *BCA*, the point *H* falls at some point, as *F*, between the points *B* and *A*. The arc *BF*, or its equal, *EH*, being equal to only a part of *BA*, is, of course, less than that arc.

Conclusion: The circles whose centers are *C* and *O*, etc.

Corollary I. In the same circle, the greater chord subtends the greater arc.

PROPOSITION VIII.

253. Theorem: ***In equal circles, the greater arc is subtended by the greater chord.***



Statement: Let the circles whose centers are *C* and *O* be equal, but let the arc *1 2* be greater than the arc *3 4*. The chord *1 2* is to be proved greater than the chord *3 4*.

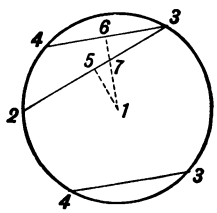
Demonstration: Suppose the chord *1 2* is not greater than the chord *3 4*. If the chord *1 2* is not greater than the chord *3 4*, it is equal to *3 4*, or less than *3 4*. The chord *1 2*, however, can be neither equal to nor less than *3 4*; for, if it were equal to *3 4*, the arcs *1 2* and *3 4* would be equal (PROP. III.); if the chord *1 2* were less than the chord *3 4*, the arc *1 2* would be less than the arc *3 4*. (PROP. VII.) Each of these conclusions is, however, contrary to the given hypothesis that the arc *1 2* is greater than the arc *3 4*. The supposition that leads to such conclusion is to be abandoned. The chord *1 2* is, then, greater than the chord *3 4*.

Conclusion: The circles whose centers are *C* and *O*, etc.

Corollary I. In the same circle, the greater arc is subtended by the greater chord.

PROPOSITION IX.

254. Theorem: ***The greater of any two chords in a circle is at the less distance from the center.***



Statement: Let *2 3* and *3 4* be any two chords of any circle. Prove *2 3* to be nearer the center than *3 4*.

Construction: If the chords are not drawn from the same point, draw from *3*, one of the extremities of the greater chord, a chord, *3 4*, equal to the less chord, and at the same distance from the center. (PROP. IV.) Draw, also, *1 5* and *1 6* from the center perpendicular to *2 3* and *3 4*.

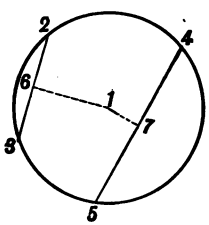
Demonstration: The less chord has the less arc, so that the point *4* falls short of the point *2*. (PROP. VII.) *1 6*, the distance of the less chord, is greater than *1 7* (AXIOM II.); and *1 7*, being an oblique line, is greater than the perpendicular *1 5*. (BOOK I., PROP. XXI.) From the continued inequality $1\,6 > 1\,7 > 1\,5$, it is evident that *1 5*, the distance of the greater chord, *2 3*, is less than *1 6*, the distance of the less chord, *3 4*.

Conclusion: The circle whose center is *1* being, etc.

Corollary I. In equal circles, the greater chord, etc.

PROPOSITION X.

255. Theorem: ***Of two chords in a circle, the chord at the less distance from the center is the greater chord.***



Statement: Let the circle whose center is *1* be any circle, and let *2 3* and *4 5* be any two of its chords, of which *4 5* is at the less distance *1 7*, from the center. *4 5* is the greater chord.

Demonstration: Suppose that the chord *4 5* is not greater than the chord *2 3*. If *4 5* is not greater than *2 3*, it is equal to *2 3*, or less than *2 3*. If the chord *4 5* be equal to the chord *2 3*, they are at equal distances from the center. (PROP. IV.) If the chord *4 5* be less than the chord *2 3*, the chord *4 5* is at the greater distance from the center. (PROP. IX.) As each of these conclusions is contrary to the hypothesis, the supposition that leads to them is to be abandoned. It follows, then, that the chord *4 5*, at the less distance from the center, is the greater chord.

Conclusion: The circle whose center is *1* being any, etc.

Corollary I. Of two chords, in equal circles, the chord at the less distance from the center is the greater chord.

MISCELLANEOUS EXERCISES.

256. 1. If two straight lines cut each other at right angles, the sums of the squares of opposite lines joining their extremities will be equal.

2. If two points be taken in the diameter of a circle, equidistant from the center, the sum of the squares of the two lines drawn from these points to any point in the circumference will be constant.

3. From a given point without a circle to draw a straight line to cut it, and to terminate in the circumference, such that the intercepted chord may have a given length, not greater than the diameter of the circle.

4. Through a given point within a circle, draw the shortest possible chord.

5. If a circle be inscribed in a triangle, the distance from the vertex of any angle to the points of contact of its sides is equal to the semi-perimeter, minus the side lying opposite to this angle.

6. The shortest line that can be drawn from a given point to a given circumference of a circle passes, when produced, through the center of the circle.

7. The longest line that can be drawn from a given point to a given circumference of a circle passes through the center of the circle.

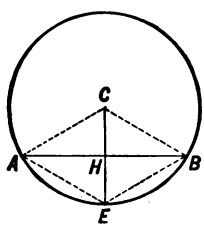
8. The radius of a circle inscribed in an equilateral triangle is one third of the altitude of the triangle.

9. The radius of a circle circumscribed about an equilateral triangle is equal to twice the distance of the sides of the triangle from the center of the circle.

SECTION III.—CHORDS AND PERPENDICULARS.

PROPOSITION XI.

257. Theorem: ***A radius perpendicular to a chord of a circle bisects that chord and also the arc subtended by the chord.***



Statement: Let the circle whose center is ***C*** be any circle, and ***A B***, any of its chords. If the radius ***C E*** is perpendicular to ***A B*** at ***H***, it bisects ***A B*** at ***H*** and the arc ***A E B*** at ***E***.

Construction: Draw ***C A*** and ***C B***, and ***A E*** and ***E B***.

Demonstration: Since ***C A*** and ***C B*** are equal oblique lines, ***A H*** and ***H B*** are equal (BOOK I., PROP. XXIV.); that is, the perpendicular ***C E*** bisects the chord at ***H***.

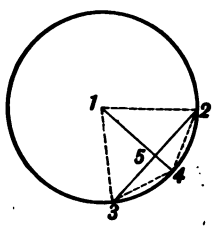
Therefore, ***A E*** and ***E B*** are equal (BOOK I., PROP. XV.); and, since they are equal, their arcs, ***A E*** and ***E B***, are equal. (PROP. III., COR. I.) The arc ***A E B*** is, then, bisected at ***E***.

Conclusion: The circle whose center is ***A*** being, etc.

Corollary I. The center of a circle and the middle points of any arc and its chord are in one and the same straight line.

PROPOSITION XII.

258. Theorem: ***A radius that bisects a chord, or that bisects the arc of a chord, is perpendicular to that chord.***



Statement: Let the circle whose center is *1* be any circle; *2 3*, any one of its chords; and *1 4*, a radius of the circle bisecting the chord, or the arc, *2 3*. The radius *1 4* is perpendicular to the chord *2 3*.

Construction: Draw the radii *1 2* and *1 3*, and the chords *3 4* and *4 2*.

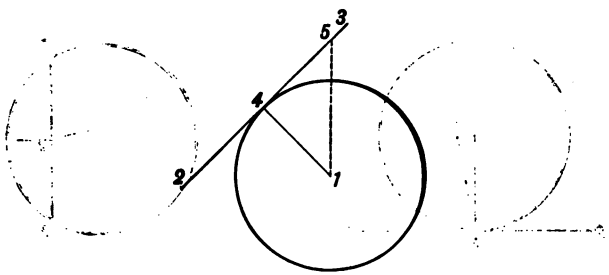
Demonstration: I. Since *5* is the middle point of *2 3*, and *1 3* and *1 2* are equal, being radii, *1 5*, or the radius *1 5 4*, is perpendicular to *2 3*, the chord which it bisects. (BOOK I., PROP. XV., COR. III.)

II. If *4* is the middle point of the arc *2 3*, the chords *2 4* and *3 4* are equal. (PROP. III., COR. I.) Since *1 2* and *1 3*, being radii, are also equal, *1 4* is perpendicular to *2 3* (BOOK I., PROP. XV., COR. II.); that is, the radius is perpendicular to the chord of the arc which it bisects.

Conclusion: The circle whose center is *1* being any circle, and *2 3* being any one of its arcs, or chords, it follows, etc.

PROPOSITION XIII.

259. Theorem: ***A straight line perpendicular to a radius at its outer extremity is tangent to the circle at that point; and, conversely, a tangent is perpendicular to the radius drawn to the point of contact.***



Statement: Let the circle whose center is *1* be any circle. If *2 3* is perpendicular to the radius *1 4* at *4*, it is tangent to the circle at *4*; and, conversely, if *2 3* is a tangent at *4*, it is perpendicular to the radius *1 4*.

Construction: Draw the straight line *1 5* to any point in *2 3*, other than the point *4*.

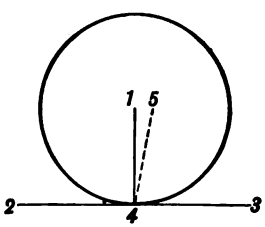
Demonstration: I. If *2 3* is perpendicular to *1 4*, *1 4* is shorter than *1 5*. (BOOK I., PROP. XXI.) As *4* is in the circumference *5*, any other point in *2 3* is beyond the circumference. *2 3* touches the circumference at but one point, and is, then, a tangent.

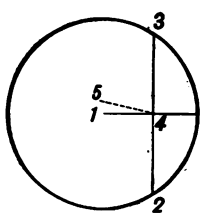
II. If *2 3* is tangent at *4*, every point of *2 3*, other than *4*, being beyond the circumference, *1 4* is the shortest line from *1* to *2 3*. It is, then, perpendicular to *2 3*. (BOOK I., PROP. XXI.)

Conclusion: The circle whose center is *1* being, etc.

PROPOSITION XIV.

260. Theorem: ***A straight line perpendicular to a tangent of a circle at the point of contact, or perpendicular to a chord at its middle point, passes through the center of the circle.***





Statement: Let the circle whose center is *1* be any circle; and let *2 3* be either any tangent or any chord, with *1 4* drawn perpendicular to the line, if a tangent, at the point of contact, or, if a chord, at the point of bisection. The straight line *1 4* passes through the center of the circle.

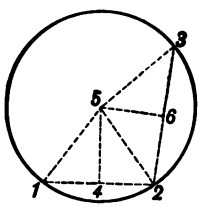
Demonstration: If the center of the circle is not in the line *1 4*, but at some point without that line, as at *5*, draw *5 4* to the point of contact or of bisection. *5 4* is perpendicular to *2 3*. (PROPS. XII. and XIII.)

This, however, is impossible, as *1 4* is perpendicular to *2 3*. (BOOK I., PROP. XI.) The supposition that leads to this conclusion is to be abandoned. The perpendicular *1 4*, then, does contain, or pass through, the center.

Conclusion: The circle whose center is *1* being any circle, and *2 3* being any tangent or any chord, it follows, etc.

PROPOSITION XV.

261. Theorem: ***Through any three points not in the same straight line, one circumference of a circle, and but one, can be described.***



Statement: Let *1, 2,* and *3* be any three points not in the same straight line. One circumference of a circle, and but one, can be drawn through these three points.

Construction: Draw *1 2* and *2 3;* and *4 5* and *6 5,* respectively, perpendicular to them at their middle points.

Demonstration: *4 5* and *6 5* must meet, otherwise they are parallel; and *2 4* and *2 6* are, then, parallel (BOOK I., PROP. XIV., COR.), and one and the same straight line (POST. III.), contrary to the hypothesis. The lines *4 5* and *5 6,* therefore, intersect, or meet at some point, as at *5.*

Since *5* is the intersection of the loci, *4 5* and *6 5,* it is the only point equally distant from *1, 2,* and *3.* (BOOK I., PROP. XV., COR. I.) The point *5* is, then, the only point equally distant from *1, 2,* and *3;* and, hence, one, and but one, circumference of a circle can be described through the given points.

Conclusion: *1, 2,* and *3* being any three points not, etc.

MISCELLANEOUS EXERCISES.

PROBLEMS.

262. I. 1. Find the locus of the points of bisection of the equal chords of a circle.

2. Having given the radius of a circle, determine its center when the circle touches two given lines which are not parallel.

3. A trapezoid may be inscribed in a circle, if the sides joining the parallel sides are equal.

4. Circumscribe a circle about a given rectangle.

5. With a given radius, describe a circle which shall pass through two given points.

II. 6. Construct a circle, the circumference of which shall pass through a given point, and be tangent to a given straight line in a given point.

7. Inscribe a circle in a given rhombus.

8. Construct a right-angled triangle, given the radius of the inscribed circle and one of the sides containing the right angle.

9. Construct a rectangle equivalent to a given rectangle and the sum of two adjacent sides equal to a given line.

10. Inscribe a circle in a given triangle.

11. Find a point in the prolongation of a diameter of a circle, from which, if a line be drawn to touch the circle, it shall be equal to a given straight line.

12. Construct a circle which shall pass through a given point, have a given radius, and be tangent to a given straight line.

III. 13. Draw a straight line tangent to two given circles.

14. Through a given point, either within or without a given circle, draw a straight line, the part of which intercepted by the circle shall be equal to a given line, not greater than the diameter of the circle.

15. Construct a circle which shall pass through two given points, and be tangent to a given straight line.

16. Two circles being given in position and magnitude, draw a straight line cutting them, so that the chords in each circle may be equal to a given line, not greater than the diameter of the smaller circle.

17. Construct a circle, the center being in the perpendicular of a given right-angled triangle, the circumference passing through the vertex of the right angle, and tangent to the hypotenuse.

18. Construct a circle having a given radius, having its center in a given straight line, and being tangent to another given straight line inclined at a given angle to the former.

19. In the diameter of a circle, produced, determine a point from which a tangent drawn to the circumference shall be equal to the diameter.

THEOREMS.

263. I. 20. In any quadrilateral figure circumscribing a circle, the opposite sides are equal to half the perimeter of the quadrilateral.

21. If from a point without a circle two secants be drawn, making equal angles with the line joining the given point and the center, the intercepted chords are equal.

22. If a circle be inscribed in a right-angled triangle, the difference between the two sides containing the right angle and the hypotenuse is equal to the diameter of the circle.

II. 23. If any number of parallel chords are drawn in a circle, their middle points lie in the same straight line.

24. Two tangents that meet are equal ; and the bisectrix of the included angle passes through the center of the circle.

25. The perpendiculars to the sides of an inscribed quadrilateral, at their middle points, meet in a common point.

26. In an inscribed trapezoid, the non-parallel sides are equal ; and, also, the diagonals.

27. Two chords which cut each other, and make equal angles with the diameter passing through their point of intersection, are equal.

III. 28. If a point be taken without a circle, and from it tangents be drawn to the circle, and another point be taken in the circumference between the two tangents, and a tangent be drawn to it, the sum of the sides of the triangle thus formed is equal to the sum of the two tangents.

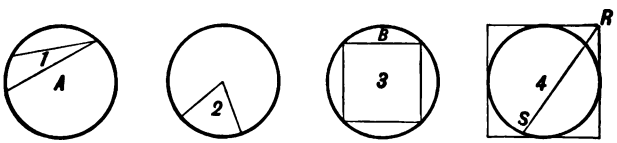
29. If any two chords of a circle intersect at right angles, the squares of their four segments will together be equivalent to the square of the diameter.

30. If a quadrilateral be circumscribed about a circle, the angles subtended at the center by one set of opposite sides will together be equal to those subtended by the other set ; that is, to two right angles.

31. If, from any point in the diameter of a semicircle, two straight lines be drawn to the circumference, one to the middle of the arc, the other at right angles to the diameter, the squares upon these two lines always amount to the same sum, wherever the point be taken.

32. In a circumscribed trapezoid, the straight line joining the middle points of the non-parallel sides is equal to one-fourth the perimeter of the trapezoid.

SECTION IV.—ANGLES AND ARCS.



DEFINITIONS.

264. An **inscribed angle** is an angle whose vertex is in the circumference of a circle and whose sides are chords; as the angle *1*.

265. An **inscribed figure** is a figure each of whose sides is a chord; as the square in the circle whose center is *3*.

266. A **circumscribed figure** is a figure each of whose sides is a tangent of a circle; as the square about the circle whose center is *4*.

267. A **sector** of a circle is a portion included between two radii and the arc subtending their included angle; as *2*.

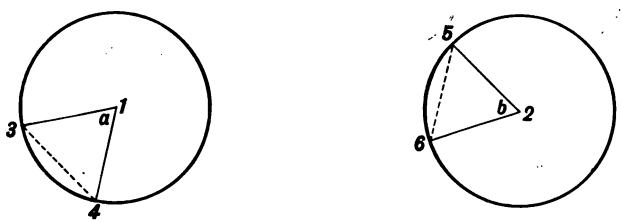
268. A **segment** of a circle is a portion included between any chord and either of its arcs; as *A* and *B*.

Point to other segments. Construct circles and divide them into segments. Construct segments without completing the circles.

269. A **secant** of a circle is a straight line which meets the circumference at two points, cutting it at one or both of the points; as the secant *R S*.

PROPOSITION XVI.

270. Theorem: ***In equal circles, equal angles at the center are subtended by equal arcs; and, conversely, equal arcs subtend equal angles at the center.***



Statement: Let the circles whose centers are *1* and *2* be any two equal circles. If the angles *a* and *b* are equal, the arcs *3 4* and *5 6* are equal; and, conversely, if the arcs *3 4* and *5 6* are equal, the angles *a* and *b* are equal.

Construction: Draw the chords *3 4* and *5 6*.

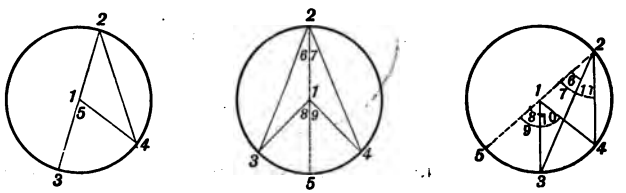
Demonstration: I. The triangles *1 3 4* and *2 5 6*, having two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, are equal in all their parts. *3 4* and *5 6* are, then, equal, and, being equal chords, the arcs *3 4* and *5 6* are equal. (PROP. III.)

II. If the arcs *3 4* and *5 6* are equal, the chords *3 4* and *5 6* are, also, equal. (PROP. II.) The triangles *1 3 4* and *2 5 6* are, then, equal, as they have their sides respectively equal. (BOOK II., PROP. III.) The angle *a* is, then, equal to the angle *b*.

Conclusion: The circles whose centers are *1* and *2*, etc.

PROPOSITION XVII.

271. Theorem: ***If an angle at the center of a circle and an angle at the circumference are subtended by the same arc, the angle at the circumference is equal to half of the angle at the center.***



Statement: Let the circle whose center is *1* be any circle. If the angles *2* and *3 1 4* have the same arc, *3 4*, the angle *2* is equal to one-half of *3 1 4*. The center of the circle may be in one of the sides of the angle *2*; or, between or beyond the sides of the angle *2*, as in the diagram.

Construction: In the second and third cases, draw *2 1*; produce it to the circumference at *5*.

Demonstration: I. The angles *2* and *4* are equal (BOOK II., PROP. VI.); and, together, they equal the angle *5*. (BOOK II., PROP. X.) The angle *2* is, then, equal to half of the angle *5*.

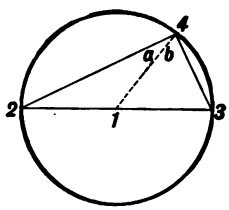
II. The angles *6* and *7* are, similarly, proved equal to halves of the angles *8* and *9*. The angle *2* is, then, equal to half of the angle *3 1 4*. (AXIOM III.)

III. The angles *6* and *7* are, similarly, proved equal to halves of the angles *8* and *9*. By equal subtractions, the angle *11* is equal to one-half of the angle *10*. (AXIOM IV.)

Conclusion: The circle whose center is *1* being, etc.

PROPOSITION XVIII.

272. Theorem: ***All the angles inscribed in semicircles are right angles.***



Statement: Let the circle whose center is *1* be any circle; let *2 3* be any of its diameters. The angle *4*, any angle inscribed in the semicircle, is a right angle.

Construction: Draw the radius *1 4*.

Demonstration: The angle *4* is equal to the sum of the angles *2* and *3*, as the parts of *4* are respectively equal to *2* and *3*. (BOOK II., PROP. VI.) The angles *4*, *2*, and *3* are together equal to two right angles. (BOOK II., PROP. XVI.) As the angle *4* equals the sum of the angles *2* and *3*, it is equal to one-half of two right angles, or one right angle.

Conclusion: As the circle whose center is *1* is, etc.

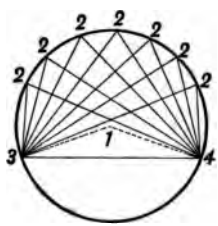
Exercises. 1. The hypotenuse being given, find the locus of the opposite vertex.

2. The hypotenuse and one side of a right triangle being given, construct the triangle.

3. The hypotenuse and its median being given, construct a right triangle.

PROPOSITION XIX.

273. Theorem: ***All the angles inscribed in a given segment of a circle, greater than a semicircle, are equal acute angles.***



Statement: Let the circle whose center is *1* be any circle, and let *3 2 4* be any segment greater than a semicircle. The inscribed angles, as the angles at the points numbered *2*, are equal acute angles.

Construction: Draw the radii *1 3* and *1 4*.

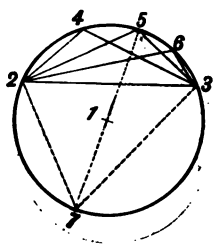
Demonstration: The inscribed angles and the angle at the center are subtended by the same arc, *3 4*. The inscribed angles are, therefore, each equal to one-half of the angle *1*. (PROP. XVII.) The angle *1*, being one of the angles of the triangle *1 3 4*, is less than two right angles. (BOOK II., PROP. XVII.) The angles *2*, *2*, *2*, etc., being each equal to half of the angle *1*, are, therefore, each less than one right angle. The inscribed angles are, then, equal acute angles.

Conclusion: The circle whose center is *1* being, etc.

Exercise. Construct angles: 90°; 60°; 45°; 30°; 150°; 135°; 120°.

PROPOSITION XX.

274. Theorem: ***All the angles inscribed in any segment of a circle, less than a semicircle, are equal obtuse angles.***



Statement: Let the circle whose center is *1* be any circle, and let the segment *2 5 3* be any of its segments less than half of the circle. All the angles inscribed in the segment, as the angles *4*, *5*, and *6*, are equal obtuse angles.

Construction: From any vertex in the segment, as *5*, draw a diameter, *5 1 7*; and draw the chords *2 7* and *3 7*.

Demonstration: In the quadrilateral *2 5 3 7*, the four angles are together equal to four right angles. (BOOK III., PROP. I.) Two of these angles, the angles at *2* and *3*, are right angles. (PROP. XVIII.) The angles *5* and *7* are, therefore, supplementary, or, together, equal to two right angles. (AXIOM IV.)

Any other angle in the segment *2 5 3* has, similarly, its supplementary angle in *2 7 3*.

The angles in *2 7 3* are, however, all equal acute angles (PROP. XIX.); and, hence, the angles inscribed in the segment *2 5 3* are all equal obtuse angles.

Conclusion: The circle whose center is *1* being, etc.

Corollary I. All angles inscribed in the same segment of a circle are equal.

MISCELLANEOUS EXERCISES.

THEOREMS.

275. I. 1. Prove that the interior angles of a hexagon are equal to eight right angles.

2. If a tangent is drawn to a circle at the extremity of a chord, the middle point of the subtended arc is equally distant from the chord and the tangent.

3. If a circle is inscribed in a right-angled triangle, the sum of its diameter and the hypotenuse is equal to the sum of the other two sides of the triangle.

4. Two chords perpendicular to a third chord at its extremities are equal.

5. The circle described on one of the equal sides of an isosceles triangle as a diameter bisects the base.

II. 6. The bisectrices of any number of angles inscribed in the same segment, pass through a common point.

7. The straight line drawn from the center of a circle to the point of intersection of two tangents bisects at right angles the chord joining their points of contact.

8. If a circle be inscribed in a right-angled triangle, and another be circumscribed about it, prove that the sum of the sides containing the right angle will be equal to the sum of the diameters.

9. If the arc of a semicircle be trisected, and from the points of section lines be drawn to either extremity of the diameter, the difference of the two segments thus made will be equal to the sector which stands on either of the arcs.

10. If the base of a triangle be produced both ways so that each part produced may be equal to the adjacent side, and through the extremities of the parts produced and the vertex a circle be described, the line joining its center and the vertex of the triangle will bisect the angle at the vertex.

11. A straight line drawn from the vertex of an equilateral triangle inscribed in a circle, to any point in the opposite circumference, is equal to two lines together which are drawn from the extremities of the base to the same point.

12. Of all triangles on the same base and between the same parallels, the isosceles has the greatest vertical angle.

III. 13. The vertical angle of a triangle is greater, or less, than a right angle, by the angle contained by the base and that diameter of the circumscribing circle which is drawn from the extremity of the base.

14. If an equilateral triangle be inscribed in a circle, the square described on a side thereof is equal to three times the square described upon the radius.

15. If, from the extremities of the diameter of a circle, tangents be drawn, and produced to intersect any other tangent, the straight lines joining the points of intersection and the center of the circle form a right angle.

16. If, from any point without a circle, lines be drawn tangent to the circle, the angle contained by the tangents is double the angle contained by the line joining the points of contact and the diameter drawn through one of them.

PROBLEMS.

276. I. 17. Draw a straight line tangent to a given circle, and making, with a given line, a given angle.

18. If either set of opposite angles of a quadrilateral be equal to two right angles, a circle may be circumscribed about the quadrilateral.

19. Draw a line tangent to a given circle and perpendicular to a given straight line.

20. Draw a line tangent to a given circle and parallel to a given straight line.

II. 21. Given the base and vertical angle of a triangle, find the locus of the vertex.

22. Find the locus of the vertices of right-angled triangles having the same hypotenuse.

23. Through a given point without a straight line, draw a straight line which shall make with the first a given angle.

24. Draw, in a circle, a chord of given length, and parallel to a given straight line.

25. With a given radius, describe a circle tangent to two given straight lines.

26. Construct a regular hexagon, one side being given.

27. On a given straight line, construct a segment that shall contain a given angle.

III. **Construct triangles, the following parts being given:**

28. The vertical angle, the line bisecting the base, and the angle which the bisecting line makes with the base.

29. The line bisecting the vertical angle, the line bisecting the base, and the difference of the angles at the base.

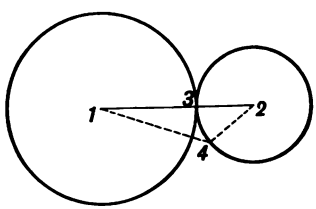
30. The vertical angle, the line drawn to the base bisecting the angle, and the difference between the base and the sum of the sides.

31. The vertical angle, the difference of the sides containing it, and the difference of their projections on the base.

SECTION V.—TWO CIRCLES.

PROPOSITION XXI.

277. Theorem: ***If the distance between the centers of two circles is equal to the sum of their radii, the circles are tangent externally.***



Statement: Let *1 2*, the distance between the centers of any two circles, be equal to the sum of the radii. The circles are tangent externally.

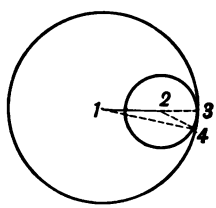
Construction: Draw the straight line *1 3 2*, the parts *1 3* and *3 2* being the radii. If the circumferences can have any other point than *3* in common, let *4* be that point; and draw *1 4* and *2 4*.

Demonstration: The point *3* is obviously common to the two circumferences. If *4* is also common, *1 4* and *2 4* are radii of the two circles; and their sum is greater than *1 2*. (BOOK II., PROP. XI.) This, however, is contrary to the hypothesis. The circumferences have, then, but one point in common, and are, therefore, tangent; and, as each center lies without the other circle, they are tangent externally.

Conclusion: The circles whose centers are *1* and *2*, etc.

PROPOSITION XXII.

278. Theorem: ***If the distance between the centers of two circles is equal to the difference of the radii, the circles are tangent internally.***



Statement: Let *1 2*, the distance between the centers of any two circles, equal the difference of the radii. The circles are tangent internally.

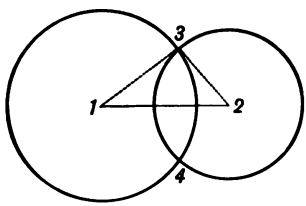
Construction: Draw *1 2*, and produce it to *3*, a point in the circumference of the circle whose center is *2*.

Demonstration: If *1 2*, the difference of the radii, be added to *2 3*, the radius of one of the circles, the sum, the straight line *1 2 3*, is equal to the radius of the other circle. The circumferences, have, then, one point, *3*, in common. If they can have another point in common, let *4* be that point. *1 4* and *2 4* are, then, the radii of the two circles, and their difference is less than the line *1 2*. (BOOK II., PROP. XI.) This, however, is contrary to the hypothesis. The circumferences have, then, but one point in common, and they are, therefore, tangent. As the center of one circle lies within the other circle, they are tangent internally.

Conclusion: As the circles whose centers are *1* and *2* are, etc.

PROPOSITION XXIII.

279. Theorem: ***If the circumferences of two circles intersect, the distance between the centers is less than the sum and greater than the difference of the radii.***



Statement: Let the circles whose centers are *1* and *2* be any two circles whose circumferences intersect, as at *3* and *4*. Prove that the distance between the centers, *1* and *2*, is less than the sum and greater than the difference of the radii.

Construction: Draw the radii *1 3* and *2 3*.

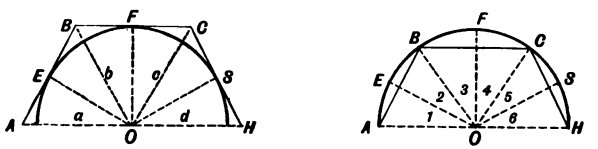
Demonstration: The distance between the centers, *1 2*, is one side of the triangle *1 3 2*. It is, then, less than the sum and greater than the difference between *1 3* and *2 3*, the radii of the two circles. (BOOK II., PROP. XI.)

Conclusion: The circles whose centers are *1* and *2*, etc.

Corollary I. If the circumferences of two circles intersect, the straight line joining the centers bisects at right angles the straight line joining the points of intersection.

PROPOSITION XXIV.

280. Theorem: *A circle may be inscribed in, or circumscribed about, any regular polygon.*



Statement: Let *A B C H* be a portion of any regular polygon. A circle may be inscribed in, or circumscribed about the polygon.

Construction: Bisect two of the sides, as *A B* and *B C*, by perpendiculars meeting at *O*. (PROP. XV.) From *O* draw straight lines to the extremities and middle points of the other sides.

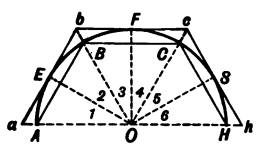
Demonstration: All the small triangles into which the regular polygon is divided are equal. (BOOK II., PROPS. I. and IV.) The lines from *O* to the vertices are, then, equal; and, the lines from *O* bisecting the sides are equal perpendiculars.

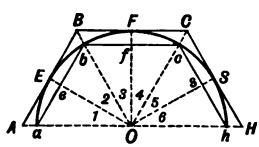
If, with *O* as a center and the distance to any vertex as a radius, a circumference be described, it will pass through all the other vertices and be circumscribed about the polygon; or if, with *O* as a center and the distance to the middle point of any side as a radius, a circumference be described, it will pass through the middle points of all the other sides; and, as the sides are then perpendicular to radii at their outer extremities, they are tangents to the circle (BOOK IV., PROP. XIII.), and the circle is inscribed within the regular polygon.

Conclusion: As *A B C H* is a portion of any, etc.

PROPOSITION XXV.

281. Theorem: ***If a regular polygon of any number of sides be inscribed in a circle, a regular polygon of the same number of sides may be circumscribed about the circle; or, if a regular polygon of any number of sides be circumscribed about a circle, a regular polygon of the same number of sides may be inscribed in the circle.***





Statement: Let *A B C H* be a portion of any regular polygon inscribed in, or circumscribed about the circle whose center is *O*. A regular polygon of the same number of sides may be circumscribed about the circle to correspond with the given inscribed polygon, or inscribed in the circle to correspond with the given circumscribed polygon.

Construction: Draw *O A*, *O B*, *O C*, and *O H* to the vertices; also, the radii, *O E*, *O F*, and *O S*, perpendicular to the sides of the given polygon. (PROP. XIII.) To obtain the corresponding circumscribed polygon, draw *a b*, *b c*, and *c h* perpendicular to the radii *O E*, *O F*, and *O S* at their outer extremities; to obtain the corresponding inscribed polygon, draw *a b*, *b c*, and *c h* so as to join the points where the lines drawn to the vertices;—viz., *O A*, *O B*, *O C*, and *O H*—intersect *the circumference.*

Demonstration I.: The sides of the given inscribed polygon being equal chords, they are at equal distances from the center. (PROP. IV.) The small triangles into which the given inscribed polygon is divided are, then, all equal. (BOOK II, PROP. I.) The small angles at the center, as *1*, *2*, *3*, etc., are thus all proved to be equal; and, the triangles *a O E*, *b O E*, *b O F*, *c O F*, *c O S*, and *h O S* are all equal. (BOOK II., PROP. II.) The figure *a b c h*, etc., is, then, a regular polygon, as all of its sides and angles are equal. The angles at *E* being equal, each is a right angle, and *a b* is, then, a tangent. (PROP. XIII.) Similarly, *b c* and *c h* are shown to be tangents. The figure *a b c h*, etc., is, then, a regular circumscribed polygon, and it has the same number of sides as the given polygon.

Demonstration II.: The sides of the given circumscribed polygon being tangents, the angles at *E*, *F*, and *S* are right angles. (PROP. XIII.) The angles at the vertices of the given polygon have been bisected, by construction; thus, the angles at *B* are equal. (BOOK II., PROP. IV.)

The series of triangles, *A O B*, *B O C*, *C O H*, etc., are, then, all equal. (BOOK II., PROP. I.) The angles at the center, *A O B*, *B O C*, *C O H*, etc., are, then, equal.

The series of triangles *a O b*, *b O c*, *c O h*, etc., are, then, equal, as their sides are equal radii. (BOOK II., PROP. I.)

The figure *a b c h*, etc., is, then, a regular polygon, as all of its sides and angles are equal. It is an inscribed polygon, as the sides are chords of the circle, and they are, moreover, the same in number as the sides of the given circumscribed polygon.

Conclusion: *A B C H* being a portion of any regular polygon, it follows, etc.

MISCELLANEOUS EXERCISES.

PROBLEMS.

282. I. 1. With a given radius, describe a circle tangent to a given straight line and given circle.

2. With a given radius, describe a circle tangent to two given circles.

3. Construct three circles having equal diameters, and being tangent to one another.

II. 4. Construct a circle which shall touch two given straight lines, and pass through a given point between them.

5. Describe a circle which shall pass through a given point and be tangent to two given circles.

6. Describe a circle which shall be tangent to two given straight lines and to a given circle.

7. Draw two concentric circles, such that the chords of the outer circle tangent to the inner circle may be equal to the diameter of the inner circle.

III. 8. Inscribe a circle in a given sector.

9 From the vertices of a triangle as centers, describe three circles which shall touch each other, two and two.

10. Draw a straight line cutting two concentric circles, so that the part intercepted by the circumference of the greater may be double the part intercepted by the less.

THEOREMS.

283. I. 11. If a straight line is a tangent to the interior of two concentric circles, and a chord in the outer, it will be bisected at the point of contact.

12. If a circle is described on the radius of another circle as a diameter, any chord of the greater, passing through the point of contact of the circles, is bisected by the circumference of the smaller.

13. Prove that the shortest distance between two circumferences is measured on the line which joins the centers.

II. 14. If two circles cut each other, and from either point of intersection diameters be drawn, their extremities and the other point of intersection are collinear.

15. If a straight line is drawn through the point of contact of two circles which touch each other, terminating in their circumferences, the radii drawn to its extremities are parallel; and, also, the tangents.

16. If two circles intersect, their common chord, produced, bisects the parts of their common tangents included between the points of contact.

17. If the sides of an equiangular and equilateral pentagon be produced to meet, the angles formed by these lines are equal, and their sum is equal to two right angles.

18. The three common chords of three circles which intersect each other, two and two, meet each other in one point.

III. 19. The square and the rhombus are the only parallelograms in which a circle can be inscribed.

20. The straight line bisecting any angle of a triangle cuts the circumference of the circumscribing circle in a point which is equidistant from the extremities of the opposite side and from the center of the inscribed circle.

21. The entire plane space about a point can be filled, without leaving vacant intervals, by equal equilateral triangles, by equal squares, and by equal regular hexagons.

Loci.

284. Find the following loci: I. 22. The points at a given distance from a given point.

23. The points at a given distance from a given straight line.

24. The points equally distant from two parallel straight lines.

25. The points equally distant from the circumferences of two concentric circles.

26. The points equally distant from two intersecting straight lines.

27. The vertex of a right-angled triangle with a given hypotenuse.

28. The vertex of a triangle with a given base and a given opposite angle.

29. The middle points of equal chords of a given length.

30. The middle points of secants from a given point to a given circle.

31. The centers of circles tangent to two intersecting straight lines.

32. The points, any one of which is equidistant from the circumferences of two equal circles.

33. The points, the sum of the squares of the distances of any point of which, from the four sides of a rectangle, shall be equal to a given square.

II. 34. The middle points of the chords of a circle passing through a given point (*a*) within the circle; (*b*) on the circumference; (*c*) beyond the circumference.

35. The middle points of chords parallel to a given *straight* line.

36. The points of intersection of tangents including a given angle.

37. The centers of circles tangent to a given straight line at a given point.

38. The centers of circles tangent to a given circle at a given point.

39. The middle point of a straight line moving between the sides of a right angle.

III. 40. Center of circle with given radius and passing through a given point.

41. Center of circle with given radius and tangent to a given straight line.

MAXIMA AND MINIMA.

285. 42. Of all straight lines which can be drawn from two given points to meet on the convex circumference of a given circle, the sum of those two which make equal angles with the tangent at the point of meeting will be the least.

43. Of polygons that are isoperimetric (i.e., that have equal perimeters), and have the same number of sides, the maximum is a regular polygon. (Isosceles triangles.)

44. Of isoperimetric regular polygons, the maximum is that which has the greatest number of sides.

If one regular polygon has one side more than the other, this other may be divided into two parts, placed to make an equivalent irregular polygon of the same number of sides as given polygon.

45. Of equivalent regular polygons, that one has the least perimeter which has the greatest number of sides.

Compare each with a regular polygon constructed with less number of sides, but area equal to polygon with greater number of sides.

46. The circumference of a circle incloses a greater area than any polygon of equal perimeter. (Method of limits.)

47. The perimeter of a circle is less than the perimeter of any equivalent polygon.

NUMERICAL EXERCISES.

286. 48. Two angles of a triangle are, respectively, $41° 5' 31''$ and $37° 52' 49''$. Find the third angle.

49. How many degrees in each angle of a square?

50. How many degrees in angle of equilateral triangle?

51. How many degrees in each of the equal angles of an isosceles triangle, the angle at the vertex being $45° 45' 40''$?

52. Which angle, if any, is obtuse in the triangle whose sides are 3, 5, and 7 inches long? Whose sides are 7, 11, 15?

53. What kind of a triangle could have sides, respectively, 5, 10, and 15? 6, 12, and 24?

54. Find the projections on the hypotenuse of the sides, 7 and 14, of a right-angled triangle. Find the distance to the hypotenuse from the opposite angle.

55. The radius of a circle is 10. How far from the center is a chord, 5?

56. The radius of a circle is 5. How far from the center is a chord, 8?

57. A chord is 7 and its distance from the center is 3. What is the radius?

58. A chord is 9 and the diameter is 10. What is the distance of the chord from the center?

59. If a chord, 16 inches long, is 10 inches from the center, how far from the center is a chord, 22 inches long?

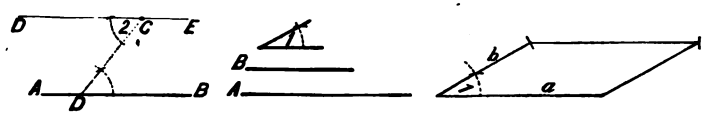
60. A circle has a radius, 9; what is the length of the *maximum* chord through a point 7 inches from the center?

CONSTRUCTIONS.

287. Prove the following constructions:

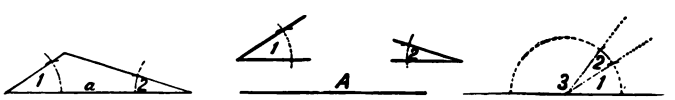
NOTE.—Prove the problems in Arts. 48, 49, 50, 60, 84, 85, 86, 121, and 186.

1. A line through a given point, parallel to a given straight line.



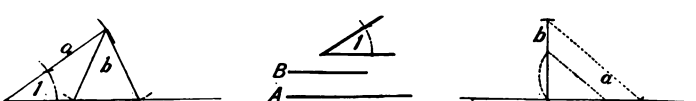
2. A parallelogram, two sides, and included angle given.

3. A triangle, two angles, and included side given.



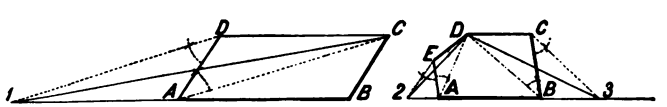
4. The third angle of a triangle, two angles given.

5. A triangle, two sides and angle opposite one given.



6. A right-angled triangle, hypotenuse, and one side given.

7. A triangle equivalent to a parallelogram.



8. A triangle equivalent to any polygon.

A triangle or parallelogram with given angle may, then, readily be constructed equivalent to any given polygon.

GENERAL SCHOLIA.

CONCYCLIC POINTS.

288. Concyclic points are points that lie in the circumference of the *same circle.*

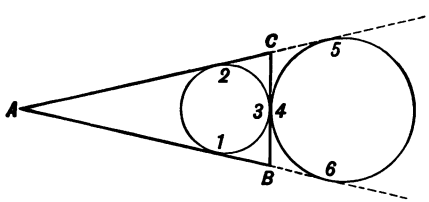
Exercises. 1. Any three points are concyclic.

2. The vertices of any regular polygon are concyclic.

3. The vertices of a quadrilateral are concyclic, if any two opposite angles are supplementary.

4. The vertices of equal given vertical angle of triangles on a given base are concyclic.

INSCRIBED AND ESCRIBED CIRCLES.



289. An **inscribed circle** is a circle tangent to the three sides of the triangle; as *1 2 3.*

290. An **escribed circle** is a circle tangent to one side of a triangle and to two other sides produced; as *4 5 6.*

291. Prove the following propositions with respect to the inscribed and any escribed circle:

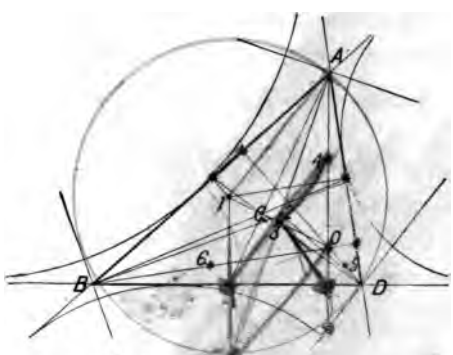
1. Tangents from the same point to a circle are equal.

2. Any tangent being a side with its production is equal to half the perimeter of the triangle; as *A 5.*

3. Any tangent being an internal segment of a side is equal to half of the perimeter diminished by the side opposite the vertex from which the tangent is drawn ; as *A 1*, or, *1 B*.

4. The intercepts between the points of contact made by any two sides of the triangle produced are equal to each other and to the third side of the triangle.

THE TRIANGLE AND NINE OF ITS CIRCLES.



The three escribed circles are represented by arcs ; the other six, by centers. Prove the construction of the nine circles. Find three other sets of concyclic points.

Circles.	Centers.	Radii.
Circumscribed,	*1*	*1 A*
Inscribed,	*2*	Perp. from *2*
Nine Points,	*3*	*3 4*
Pedal (3),	*4, 5, 6*	*4 0, 5 0, 6 0*

C is the *centroid ;* and *O*, the *orthocenter ; 13 = 30*

292. Prove the following propositions :

1. The circumscribed circle bisects the straight lines, joining the center of the inscribed circle with the centers of the escribed circles.

2. Each vertex of the triangle is collinear with the centers of two of the escribed circles.

3. The center of the inscribed circle is collinear with the center of any escribed circle and the opposite vertex.

4. Each center of the inscribed or the escribed circles is the orthocenter of the triangle, having the other three centers as its vertices. (See § 293, for definition)

5. The four circles, each of which passes through three of the centers of the escribed and inscribed circles, are equal.

6. The three circles, the circumference of each of which passes through the extremities of any side of a triangle and the orthocenter, equal one another and the original triangle.

Nineteen circles in all have been mentioned.

THE NINE-POINTS CIRCLE.

293. The **orthocenter** is the point at which the three perpendiculars from the vertices to the opposite sides meet.

294. The **centroid** of any triangle is the point at which the medians of the triangle meet.

295. Prove the following propositions:

1. The middle points of the sides of a triangle are concyclic with the feet of the perpendicular from the opposite vertices, and the middle points of the lines joining the orthocenter with the vertices. (***Nine-points circle.***)

2. The center of the nine-points circle is the middle point of the line joining the orthocenter and the center of the circumscribed circle.

3. The diameter of the nine-points circle is equal to the radius of the circumscribed circle.

4. The orthocenter and the centroid are collinear with the centers of the nine-points and circumscribed circles.

5. The nine-points circle is tangent to the inscribed and escribed circles of a triangle.

BOOK V.—PROPORTION.

SYNOPSIS.

SECTION I.—EQUAL RATIOS.

296. 1. DEFINITIONS.

2 PROPOSITIONS.

PROP. I. If four quantities are in proportion, the product of the extremes is equal to the product of the means.

PROP. II. If four quantities are in proportion, they are in proportion by alternation.

PROP. III. If four quantities are in proportion, they are in proportion by inversion.

PROP. IV. If four quantities are in proportion, they are in proportion by composition.

PROP. V. If four quantities are in proportion, they are in proportion by division.

SECTION II.—EQUAL PRODUCTS.

297. PROP. VI. If the product of two quantities is equal to the product of two other quantities, the two quantities that form either product may be made the extremes, and the other two quantities, the means, of a proportion.

PROP. VII. If the product of three quantities equals the product of three other quantities, the three that form either product may be separated in any manner to form the extremes, and the other three quantities will form the means of a proportion.

PROP. VIII. If the product of two quantities is equal to the square of a third quantity, the third quantity is a mean proportional between the two other quantities.

PROP. IX. If the product of two quantities is equal to a third quantity, the square root of the third quantity is a mean proportional between the two other quantities.

PROP. X. If the product of three quantities is equal to a fourth quantity, the square root of the fourth quantity is a mean proportional between any one and the remaining two of the three quantities.

SECTION III.—SEVERAL PROPORTIONS.

298. PROP. XI. If a couplet in each of two proportions is the same, the other couplets form a proportion.

PROP. XII. If, in two proportions, the antecedents are the same, the consequents form a proportion.

PROP. XIII. If, in two proportions, the consequents are the same, the antecedents form a proportion.

PROP. XIV. If the terms of two proportions, taken in the same order, are multiplied together, the products will form a proportion.

PROP. XV. If the terms of one proportion are divided by the terms of another proportion, taken in the same order, the quotients will form a proportion.

SECTION IV.—CONTINUED PROPORTIONS.

299. PROP. XVI. In any proportion, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

PROP. XVII. In any proportion, all the antecedents, or all the consequents, may be multiplied by any quantity, and the results will be proportional.

PROP. XVIII. In any proportion, all the antecedents, or all the consequents, may be divided by any quantity, and the results will be proportional.

PROP. XIX. The same powers of the terms of a proportion form a proportion.

PROP. XX. The same roots of the terms of a proportion form a proportion.

SECTION I.—EQUAL RATIOS.

DEFINITIONS.

300. Ratio is the relation between two quantities, expressed by division.

The ratio of 30 to 5 is $\frac{30}{5}$, or $30 \div 5$; written in abbreviated form, $30 : 5$, the ratio being equal to 6.

301. Proportion is an equality of ratios.

The ratio of 30 to 6 equals the ratio of 45 to 9, as each ratio is equal to 5. The four quantities form a proportion, written thus:

$$\frac{30}{6} = \frac{45}{9};\ \text{or,}$$
$$30 \div 6 = 45 \div 9;\ \text{or,}$$
$$30 : 6 = 45 : 9;\ \text{or,}$$
$$30 : 6 :: 45 : 9.$$

Any one of the preceding forms may be read: 30 is to 6 as 45 is to 9.

302. Proportion is the basis of mensuration. Either of two quantities of the same kind may be taken as the measure of the other, and their relation may be expressed by division, as a ratio; and, when two quantities have a common measure, there is an equality of ratios, or a proportion.

Thus, if $A = 5 \times M$, and, $B = 7 \times M$, then, $M = \frac{A}{5}$, also, $M = \frac{B}{7}$, and, $\frac{A}{5} = \frac{B}{7}$; or, $A : 5 :: B : 7$.

303. The first and last terms are the **extremes**.

In the proportion, $a : b :: c : d$, the *extremes* are a and d.

304. The second and third terms are the **means**.

In the proportion, $a : b :: c : d$, the *means* are b and c.

305. The first, second, third, and fourth terms may be called the first, second, third, and fourth **proportionals.**

When three quantities form a proportion, the second and third terms being the same quantity, it is called a **mean proportional.**

306. Any two terms forming a ratio, as the first and second terms, or the third and fourth terms, form a **couplet.**

307. The first terms in the different couplets are the **antecedents.**

In the proportion, $a : b :: c : d$, the *antecedents* are a and c.

308. The second terms in the different couplets are the **consequents.**

In the proportion, $a : b :: c : d$, the *consequents* are b and d.

309. By **alternation** is meant comparison of the first term with the third, and the second term with the fourth.

The proportion, $a : b :: c : d$, by alternation becomes $a : c :: b : d$.

310. By **inversion** is meant making, in each couplet, the first term second and the second term first.

In the proportion, $a : b :: c : d$, the terms are *inverted* by making the proportion read, $b : a :: d : c$.

311. By **composition** is meant comparing the sum of first and second terms in each couplet with the first or the second term of the couplet.

$a : b :: c : d$, by *composition*, becomes $a + b : b :: c + d : d$; or, $a + b : a :: c + d : c$.

312. By **division** is meant comparing the difference of the first and second terms in each couplet with the first or the second term of the couplet.

$a : b :: c : d$, by *division*, becomes $a - b : b :: c - d : d$; or, it becomes $a - b : a :: c - d : c$.

PROPOSITION I.*

313. Theorem: ***If four quantities are in proportion, the product of the extremes is equal to the product of the means.***

Statement: If $A : B :: C : D$, then, $AD = BC$.

Demonstration: Write the proportion in fractional form,

$$\frac{A}{B} = \frac{C}{D};$$

clearing (AXIOM V.), $AD = BC$. *Q. E. D.*

PROPOSITION II.

314. Theorem: ***If four quantities are in proportion, they are in proportion by alternation.***

Statement: If $A : B :: C : D$, then, $A : C :: B : D$.

Demonstration: $AD = BC$ (PROP. I.);

dividing by proposed consequents (AXIOM VI.),

$$\frac{AD}{CD} = \frac{BC}{CD};$$ and, canceling, $\frac{A}{C} = \frac{B}{D}$,

which is the same as $A : C :: B : D$. *Q. E. D.*

* This proposition is used as the test of a proportion. If the product of the extremes is not equal to the product of the means, the quantities are not in proportion. It is also used in finding any missing term of a proportion.

PROPOSITION III.

315. Theorem: ***If four quantities are in proportion, they are in proportion by inversion.***

Statement: If $A : B :: C : D$, then, $B : A :: D : C$.

Demonstration: $BC = AD$ (PROP. I.);

dividing by proposed consequents,

$$\frac{BC}{AC} = \frac{AD}{AC} \qquad \text{(AXIOM VI.)};$$

by canceling, $\frac{B}{A} = \frac{D}{C}$; or, $B : A :: D : C$. *Q. E. D.*

PROPOSITION IV.

316. Theorem: ***If four quantities are in proportion, they are in proportion by composition.***

Statement: If $A : B :: C : D$, then, $A + B : B :: C + D : D$.

Demonstration: $\frac{A}{B} = \frac{C}{D}$; adding 1 to each member,

$$\frac{A}{B} + 1 = \frac{C}{D} + 1 \qquad \text{(AXIOM III.)};$$

reducing, $\frac{A+B}{B} = \frac{C+D}{D}$; or,

$A + B : B :: C + D : D$. *Q. E. D.*

Corollary I. The sum of antecedent and consequent is proportional to the antecedent.

PROPOSITION V.

317. Theorem: ***If four quantities are in proportion, they are in proportion by division.***

Statement: If $A : B :: C : D$, then, $\begin{cases} A - B : B :: C - D : D; \text{ or,} \\ B - A : A :: D - C : C. \end{cases}$

Demonstration: I. Writing the proportion in fractional form, $\frac{A}{B} = \frac{C}{D}$, and, subtracting 1 from each member, $\frac{A}{B} - 1 = \frac{C}{D} - 1$; and this, by reduction, becomes $\frac{A - B}{B} = \frac{C - D}{D}$. Writing the result in proportional form, $A - B : B :: C - D : D$. *Q. E. D.*

II. By inversion, $B : A :: D : C$ (PROP. III.); or, in fractional form, the equation is,

$$\frac{B}{A} = \frac{D}{C}.$$

Subtracting 1 from both members, $\frac{B}{A} - 1 = \frac{D}{C} - 1$, which, by reduction, becomes $\frac{B - A}{A} = \frac{D - C}{C}$; or,

$$B - A : A :: D - C : C.$$ *Q. E. D.*

Corollary I. The difference between antecedent and consequent is proportional to either the antecedent or the consequent.

SECTION II.—EQUAL PRODUCTS.

PROPOSITION VI.

318. Theorem: ***If the product of two quantities is equal to the product of two other quantities, the two quantities that form either product may be made the extremes, and the other two quantities, the means, of a proportion.***

Statement: If $AD = BC$, then, $A : B :: C : D$.

Demonstration: Dividing by the proposed consequents,

$$\frac{AD}{BD} = \frac{BC}{BD} \text{ (AXIOM VI.) ; canceling, } \frac{A}{B} = \frac{C}{D};$$

or, $A : B :: C : D$. *Q. E. D.*

PROPOSITION VII.

319. Theorem: ***If the product of three quantities equals the product of three other quantities, the three that form either product may be separated in any manner to form the extremes, and the other three quantities will form the means of a proportion.***

Statement: If $ABC = DEF$, then, $A : EF :: D : BC$.

Demonstration: Divide both members by the proposed consequents, as $EFBC$;

$$\frac{ABC}{EFBC} = \frac{DEF}{EFBC}; \text{ canceling, } \frac{A}{EF} = \frac{D}{BC};$$

or, $A : EF :: D : BC$. *Q. E. D.*

PROPOSITION VIII.

320. Theorem: ***If the product of two quantities is equal to the square of a third quantity, the third quantity is a mean proportional between the two other quantities.***

Statement: If $AC = B^2$, then, $A : B :: B : C$.

Demonstration: Dividing by the proposed consequents,

$$\frac{AC}{BC} = \frac{B^2}{BC}; \text{ canceling, } \frac{A}{B} = \frac{B}{C};$$

or, $A : B :: B : C$. *Q. E. D.*

PROPOSITION IX.

321. Theorem: ***If the product of two quantities is equal to a third quantity, the square root of the third quantity is a mean proportional between the two other quantities.***

Statement: If $AC = B$, then, $A : \sqrt{B} :: \sqrt{B} : C$.

Demonstration: Dividing both members by $C\sqrt{B}$, the product of the proposed consequents,

$$\frac{AC}{C\sqrt{B}} = \frac{B}{C\sqrt{B}}; \text{ canceling, } \frac{A}{\sqrt{B}} = \frac{\sqrt{B}}{C};$$

or, $A : \sqrt{B} :: \sqrt{B} : C$. *Q. E. D.*

PROPOSITION X.

322. Theorem: *If the product of three quantities is equal to a fourth quantity, the square root of the fourth quantity is a mean proportional between any one and the remaining two of the three quantities.*

Statement: If $A\,B\,C = D$, then, $A : \sqrt{D} :: \sqrt{D} : B\,C$.

Demonstration: Dividing both members by $B\,C\,\sqrt{D}$, the product of the proposed consequents,

$$\frac{A\,B\,C}{B\,C\,\sqrt{D}} = \frac{D}{B\,C\,\sqrt{D}}; \text{ canceling, } \frac{A}{\sqrt{D}} = \frac{\sqrt{D}}{B\,C};$$

or, $A : \sqrt{D} :: \sqrt{D} : B\,C$. *Q. E. D.*

SECTION III.—SEVERAL PROPORTIONS.

PROPOSITION XI.

323. Theorem: *If a couplet in each of two proportions is the same, the other couplets form a proportion.*

Statement: If $A : B :: C : D$, and $A : B :: E : F$,

then, $C : D :: E : F$.

Demonstration: In fractional form, $\frac{A}{B} = \frac{C}{D}$ and $\frac{A}{B} = \frac{E}{F}$;

therefore (AXIOM 1.), $\frac{C}{D} = \frac{E}{F}$; or, $C : D :: E : F$. *Q. E. D.*

PROPOSITION XII.

324. Theorem: ***If, in two proportions, the antecedents are the same, the consequents form a proportion.***

Statement: If $A:B::C:D$, and $A:E::C:F$, then,

$B:D::E:F$, and $B:E::D:F$.

Demonstration: By alternation, the two given proportions become $A:C::B:D$, and $A:C::E:F$ (PROP. II.);

therefore, $B:D::E:F$ (PROP. XI.);

and, by alternation, $B:E::D:F$. *Q. E. D.*

PROPOSITION XIII.

325. Theorem: ***If, in two proportions, the consequents are the same, the antecedents form a proportion.***

Statement: If $A:B::C:D$, and $E:B::F:D$, then,

$A:C::E:F$, and $A:E::C:F$.

Demonstration: By alternation, the two given proportions become $A:C::B:D$, and $E:F::B:D$ (PROP. II.);

therefore, $A:C::E:F$ (PROP. XI.);

and, by alternation, $A:E::C:F$. *Q. E. D.*

Scholium. If any two terms of one proportion are the same as two corresponding terms of another proportion, the remaining terms form a proportion.

Proposition XIV.

326. Theorem: *If the terms of two proportions, taken in the same order, are multiplied together, the products will form a proportion.*

Statement: If $A : B :: C : D$, and $E : F :: G : H$, then,

$$AE : BF :: CG : DH.$$

Demonstration: $\frac{A}{B} = \frac{C}{D}$, and $\frac{E}{F} = \frac{G}{H}$;

multiplying, $\frac{AE}{BF} = \frac{CG}{DH}$ (Axiom v.);

or, $AE : BF :: CG : DH.$ *Q. E. D.*

Proposition XV.

327. Theorem: *If the terms of one proportion are divided by the terms of another proportion, taken in the same order, the quotients will form a proportion.*

Statement: If $A : B :: C : D$, and $E : F :: G : H$, then,

$$\frac{A}{E} : \frac{B}{F} :: \frac{C}{G} : \frac{D}{H}.$$

Demonstration: $AD = BC$, and $EH = FG$ (Prop. I.);

then, $\frac{AD}{EH} = \frac{BC}{FG}$ (Axiom VI.);

or, $\frac{A}{E} \times \frac{D}{H} = \frac{B}{F} \times \frac{C}{G}$;

and, therefore (Prop. VI.),

$$\frac{A}{E} : \frac{B}{F} :: \frac{C}{G} : \frac{D}{H}.$$ *Q. E. D.*

Scholium. If the antecedents, or the consequents, or the terms of either couplet be increased or decreased by like parts of each, the results will be proportional.

SECTION IV.—CONTINUED PROPORTIONS.

328. A **continued proportion** is a continued equality of ratios.

$A : B :: C : D :: E : F :: G : H :: I : J$, is a *continued proportion*; read, A is to B as C is to D as E is to F as G is to H as I is to J.

PROPOSITION XVI.

329. Theorem: ***In any proportion, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.***

Statement: If $A : B :: C : D :: E : F :: G : H$, then,

$$A + C + E + G : B + D + F + H :: A : B.$$

Demonstration: By identity, $AB = BA$;

and, $AD = BC$ (PROP. I.);

also, $AF = BE$ (PROP. I.);

and, $AH = BG$ (PROP. I.);

therefore, $A(B + D + F + H) = B(A + C + E + G)$ (AXIOM III.),

and (PROP. VI.), $A + C + E + G : B + D + F + H :: A : B$.

Q. E. D.

Scholium. For the second couplet in the conclusion may be substituted any other couplet in the given proportion, as ratios equal to the same ratio are equal to each other.

PROPOSITION XVII.

330. Theorem: ***In any proportion, all the antecedents, or all the consequents, may be multiplied by any quantity, and the results will be proportional.***

Statement: If $A : B :: C : D :: E : F$, then,
$MA : B :: MC : D :: ME : F$; and, $A : MB :: C : MD :: E : MF$.

Demonstration: $\frac{A}{B} = \frac{C}{D} = \frac{E}{F}$; then (AXIOM v.),

$\frac{MA}{B} = \frac{MC}{D} = \frac{ME}{F}$, and, $\frac{A}{MB} = \frac{C}{MD} = \frac{E}{MF}$, or,

$MA : B :: MC : D :: ME : F$, and, $A : MB :: C : MD :: E : MF$.

Q. E. D.

Corollary I. Any like multiples of the antecedents are proportional to any like multiples of the consequents.

PROPOSITION XVIII.

331. Theorem: ***In any proportion, all the antecedents, or all the consequents, may be divided by any quantity, and the results will be proportional.***

Statement: If $A : B :: C : D :: E : F$, then,

$\frac{A}{M} : B :: \frac{C}{M} : D :: \frac{E}{M} : F$; and, $A : \frac{B}{M} :: C : \frac{D}{M} :: E : \frac{F}{M}$.

Demonstration: $NA : B :: NC : D :: NE : F$ (PROP. XVII.),

and, also, $A : NB :: C : ND :: E : NF$; making $N = \frac{1}{M}$,

$\frac{A}{M} : B :: \frac{C}{M} : D :: \frac{E}{M} : F$, and, $A : \frac{B}{M} :: C : \frac{D}{M} :: E : \frac{F}{M}$.

Q. E. D.

PROPOSITION XIX.

332. Theorem: *The same powers of the terms of a proportion are proportional.*

Statement: If $A : B :: C : D :: E : F$, then,

$$A^m : B^m :: C^m : D^m :: E^m : F^m.$$

Demonstration: Repeating the proportion and multiplying,

$$A : B :: C : D :: E : F$$

$$A : B :: C : D :: E : F$$

$$A : B :: C : D :: E : F$$

$$A^3 : B^3 :: C^3 : D^3 :: E^3 : F^3 \qquad \text{(PROP. XIV.)};$$

and, similarly, $A^m : B^n :: C^m : D^m :: E^m : F^m$. *Q. E. D.*

PROPOSITION XX.

333. Theorem: *The same roots of the terms of a proportion are proportional.*

Statement: If $A : B :: C : D :: E : F$, then,

$$\sqrt[m]{A} : \sqrt[m]{B} :: \sqrt[m]{C} : \sqrt[m]{D} :: \sqrt[m]{E} : \sqrt[m]{F}.$$

Demonstration: $A^n : B^n :: C^n : D^n :: E^n : F^n$ (PROP. XIX.);

making $$n = \frac{1}{m},$$

$$A^{\frac{1}{m}} : B^{\frac{1}{m}} : C^{\frac{1}{m}} : D^{\frac{1}{m}} :: E^{\frac{1}{m}} : F^{\frac{1}{m}};$$ which is the

same as, $\sqrt[m]{A} : \sqrt[m]{B} :: \sqrt[m]{C} : \sqrt[m]{D} :: \sqrt[m]{E} : \sqrt[m]{F}$. *Q. E. D.*

BOOK VI.—MENSURATION: PLANE FIGURES.

SYNOPSIS.

SECTION I.—ANGLES

334. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. I. In equal circles, angles at the center are proportional to their intercepted arcs.

PROP. II. An inscribed angle is measured by half of its intercepted arc.

PROP. III. An angle formed by a tangent to a circle and a chord drawn to the point of contact is measured by half of the intercepted arc.

PROP. IV. In any circle, angles formed by intersecting chords are measured by half the sum of the intercepted arcs.

PROP. V. Angles formed by secants meeting without the circle are measured by half of the difference of the intercepted arcs.

3. MISCELLANEOUS EXERCISES.

SECTION II.—TRIANGLES.

335. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. VI. Triangles with equal altitudes are proportional to their bases.

PROP. VII. Triangles, with an angle in each equal, are proportional to the product of the including sides.

PROP. VIII. The area of a right-angled triangle is equal to half the product of its base and altitude.

PROP. IX. The area of any triangle is equal to half the product of its base and altitude.

PROP. X. A trapezium being divided into two triangles by either of its diagonals, its area is equal to half of the product of that diagonal and the sum of the perpendiculars drawn from the remaining vertices to the diagonal, or the diagonal produced.

3. MISCELLANEOUS EXERCISES.

SECTION III.—QUADRILATERALS.

336. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XI. Parallelograms with equal altitudes are proportional to their bases.

PROP. XII. Parallelograms, with an angle in each equal, are proportional to the products of the including sides.

PROP. XIII. The area of a rectangle is equal to the product of its base and altitude.

PROP. XIV. The area of a parallelogram is equal to the product of its base and altitude.

PROP. XV. The area of a trapezoid is equal to the product of its altitude and half the sum of its parallel sides.

3. MISCELLANEOUS EXERCISES.

SECTION IV.—REGULAR POLYGONS.

337. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XVI. To inscribe a regular polygon of the same number of sides as a given circumscribed polygon; or, to circumscribe a regular polygon of the same number of sides as a given inscribed polygon.

PROP. XVII. To find the length of one side and the area of any inscribed or circumscribed square.

PROP. XVIII. The side of a regular inscribed hexagon is equal to the radius of a circle.

PROP. XIX. The perimeter and apothem of a regular inscribed polygon of any number of sides being given, to find the perimeter and apothem of a regular inscribed polygon of double the number of sides.

PROP. XX. The perimeter of a regular circumscribed polygon being given, to find the length of a side of the perimeter of the regular circumscribed polygon of double the number of sides.

3. MISCELLANEOUS EXERCISES.

SECTION V.—CIRCLES.

338. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XXI. To find the circumference of a circle, from the perimeter of the inscribed and circumscribed squares.

PROP. XXII. To find the area of a circle.

PROP. XXIII. To find the area of a sector of a circle.

PROP. XXIV. To find the area of a segment of a circle.

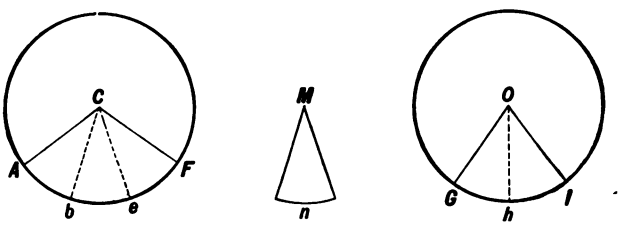
PROP. XXV. To find the area of a circular ring.

3. MISCELLANEOUS EXERCISES.

SECTION I.—ANGLES.

PROPOSITION I.

339. Theorem: ***In equal circles, angles at the center are proportional to the intercepted arcs.***



Statement: Let ACF and GOI be any angles at the centers of equal circles. Prove that $ACF : GOI :: AF : GI$.

I. Commensurable angles.

Construction: When the angles have a *common measure*, as the angle M, apply it to the angles C and O as often as it is found in each; thus, three times in C and twice in O.

Demonstration: By hypothesis, $M = \frac{ACF}{3}$, and $M = \frac{GOI}{2}$;

therefore, $\frac{ACF}{3} = \frac{GOI}{2}$; or, $ACF : GOI :: 3 : 2$. (A)

The angles at the center being equal, the arcs Ab, be, eF, Gh, and hI, are equal. (BOOK IV., PROP. XVI.)

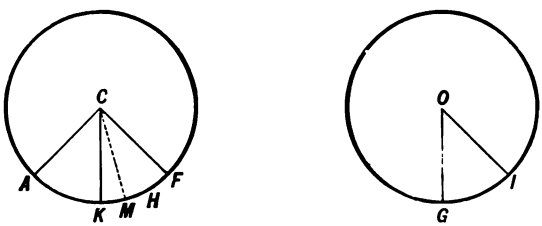
By hypothesis, then, $n = \frac{AF}{3}$, and $n = \frac{GI}{2}$;

therefore, $\frac{AF}{3} = \frac{GI}{2}$; or, $AF : GI :: 3 : 2$. (B)

Then (BOOK V. PROP. XI.), $ACF : GOI :: AF : GI$. Q. E. D.

II. Incommensurable angles.

Construction: When the angles have *no common measure*, take *A C K*, a part of the larger angle, *A C F*, equal to *G O I*.



Demonstration: The arc *A K* is equal to *G I*. (PROP. XVI.)

If the proportion, *A C F : A C K : : A F : A K*, is not true, suppose the fourth term to be *A H*, greater than *A K*; thus,

$$A C F : A C K : : A F : A H. \qquad (A)$$

Divide *A C F* into equal parts, each less than *K C H*. At least one line of division, as *C M*, falls between *C K* and *C H*. The angles *A C F* and *A C M* being commensurable,

$$A C F : A C M : : A F : A M; \qquad (B)$$

and, therefore, comparing (A) and (B),

$$A C K : A C M : : A H : A M. \quad (\text{BOOK V., PROP. XII.})$$

As *A C K* is less than *A C M*, *A H* should, then, be less than *A M*, whereas *A H* is greater than *A M*. The supposition that the fourth term can be greater than *A K* leads to a false conclusion.

In similar manner, it can be shown that the fourth term is not less than *A K*. The fourth term, therefore, is *A K*, and the proportion, *A C F : A C K : : A F : A K*, is true; and, by substitution (AXIOM I.), $A C F : G O I : : A F : G I$. *Q. E. D.*

Conclusion: The circles whose centers are *C* and *O*, etc.

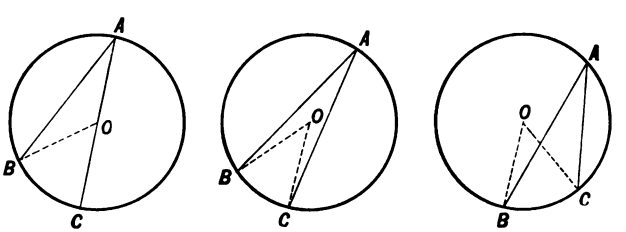
Corollary I. In any circle, angles at the center are proportional to the intercepted arcs.

Corollary II. Angles at the center of a circle are measured by their intercepted arcs.

Corollary III. All the angles at the center of a circle, being equal to four right angles, are measured by the entire circumference ; and one right angle is measured by a quadrant, or one quarter of a circumference.

PROPOSITION II.

340. Theorem: ***An inscribed angle is measured by half of its intercepted arc.***



Statement: Let the circle whose center is *O* be any circle ; *A*, any angle inscribed in the circle ; and, *B C*, its intercepted arc. The angle *A* is measured by half of *B C*.

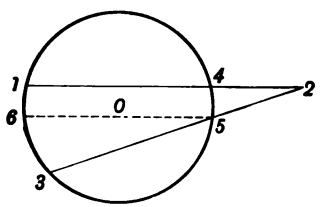
Construction: Construct the angle *B O C*, at the center.

Demonstration: The angle *B O C* is measured by the arc *B C*. (PROP. I., COR. II.) The angle *B A C* is equal to half of the angle *B O C*. (BOOK IV., PROP. XVII.) The angle *B A C* is, then, measured by half of *B C*. (AXIOM VI.)

Conclusion: The circle whose center is *O* being, etc.

PROPOSITION V.

343. Theorem: ***Angles formed by secants meeting without the circle are measured by half of the difference of the intercepted arcs.***



Statement: Let *1 2* and *2 3* be two secants intersecting without any circle, as at *2*. The angle *1 2 3* is measured by half of the difference of the intercepted arcs *1 3* and *4 5*.

Construction: From *5*, draw the chord *5 6* parallel to *1 4*.

Demonstration: The angle *2* is equal to the angle *5*. (BOOK I., PROP. XVII.) The angle *5* is measured by half of the arc *6 3*, which is the difference of the arcs *1 3* and *1 6* (PROP. II.), or its equal, *4 5*. (BOOK IV., PROP. V.) If the angle *5* is measured by half of the difference between the arcs *1 3* and *4 5*, its equal, the angle *2*, is also measured by half of the difference between *1 3* and *4 5*, the intercepted arcs.

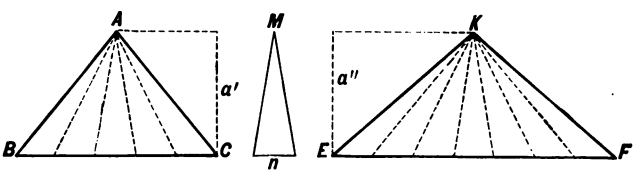
Conclusion: The circle whose center is *0* being, etc.

Corollary I. An angle formed by a tangent and a secant, or by two tangents, is measured by half of the difference of *the intercepted* arcs.

SECTION II.—TRIANGLES.

PROPOSITION VI.

344. Theorem: ***Triangles with equal altitudes are proportional to their bases.***



Statement: Let *ABC* and *KEF* be any two triangles, with altitudes, a' and a'', equal. Prove that $ABC : KEF :: BC : EF$.

I. Commensurable bases.

Construction: Taking n as the common measure of the two bases, construct any triangle, M, with the given altitude. Lay off n, the common measure, on the two bases as often as it is found in each, as five times in BC, and seven times in EF. Draw lines from the vertices, A and K, to the points of division.

Demonstration: $ABC = 5M$, and $EKF = 7M$ (BOOK III., PROP. XX., COR. II.);

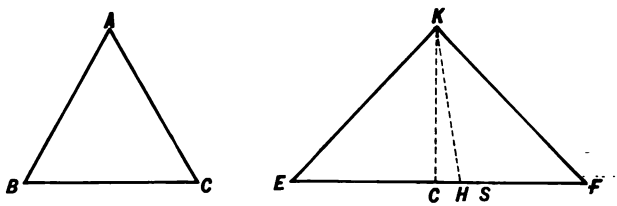
then, $M = \frac{ABC}{5} = \frac{EKF}{7}$; and, $ABC : EKF :: 5 : 7$. (A)

Moreover, $n = \frac{BC}{5}$, and $n = \frac{EF}{7}$; or, $BC : EF :: 5 : 7$. (B)

Then, comparing (A) and (B), (BOOK V., PROP. XI.),

$$ABC : EKF :: BC : EF.$$ Q. E. D.

II. Incommensurable bases.



Construction: On *EF*, lay off *EC* equal to *BC*. Draw *KC*.

Demonstration: The triangles *KEC* and *ABC* are equal. (Book III., Prop. XVII., Cor. II.) If the proportion,

$$EKF : EKC :: EF : EC,$$ is untrue, suppose
$$EKF : EKC :: EF : ES,$$

the fourth term, *ES*, being greater than *EC*. Divide *EF* into equal parts, each less than *CS*. At least one point of division, as *H*, will fall between *C* and *S*. Draw *KH*.

Since *EF* and *EH* are commensurable,

$$EKF : EKH :: EF : EH;$$

therefore, $EKC : EKH :: ES : EH$. (Book V., Prop. XII.) In this proportion, *EKC* is less than *EKH*; *ES* should, then, be less than *EH*, whereas it is greater than *EH*. The supposition that leads to this false conclusion is to be abandoned.

In similar manner, it can be shown that the fourth term cannot be less than *EC*. The first proportion is true, and

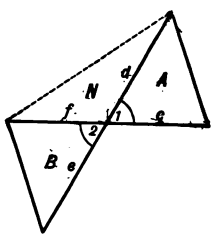
$$EKF : EKC :: EF : EC;$$ or, $EKF : ABC :: EF : BC,$

by substituting for *EKC* and *EC* their equals, *ABC* and *BC*.

Conclusion: *ABC* and *KEF* being any triangles, etc.

PROPOSITION VII.

345. Theorem: ***Triangles with an angle in each equal are proportional to the product of the including sides.***



Statement: Let the triangles *A* and *B* have the angles *1* and *2* equal. Prove that $A : B :: c \times d : e \times f$.

Construction: Place the triangles so that the equal angles are opposite, or vertical. Complete the figure *N*.

Demonstration: $A : N :: c : f$, and $N : B :: d : e$. (PROP. VI.)

$A \times N : B \times N :: c \times d : e \times f$. (BOOK V., PROP. XIV.)

$A : B :: c \times d : e \times f$. (BOOK V., PROP. XV., COR.)

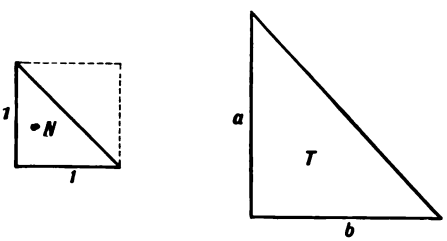
Conclusion: *A* and *B* being any triangles, etc.

Corollary I. If a triangle is divided into two triangles by a line bisecting one of its angles, the two triangles thus made are proportional to the two sides of the angle bisected.

Corollary II. The segments of the base of a triangle that are made by a line bisecting the vertical angle are proportional to the sides of the bisected angle.

PROPOSITION VIII.

346. Theorem: ***The area of a right-angled triangle is equal to half of the product of its base and altitude.***



Statement: Let T be any right-angled triangle; b, the base; and a, the altitude.

Construction: Construct N, any right-angled triangle, with its base and altitude each equal to 1, any linear unit.

Demonstration: As T and N have an angle in each equal,

$$T : N :: a \times b : 1 \times 1. \qquad \text{(PROP. VII.)}$$

In this proportion, 1×1 equals 1; and N, being a right-angled triangle, on completing the square is evidently half of 1, the square unit. (BOOK III., PROP. XVI.) Substituting $\frac{1}{2}$ for N, and 1 for 1×1, the proportion becomes

$$T : \tfrac{1}{2} :: a \times b : 1;$$

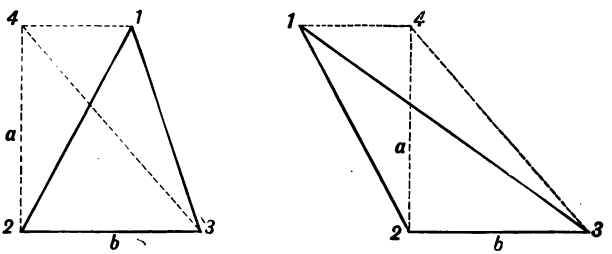
therefore,

$$T = \tfrac{1}{2}(a \times b). \qquad \text{(BOOK V., PROP. I.)}$$

Conclusion: T being any right-angled triangle, it follows, in general, that: The area of a right-angled triangle is equal to half of the product of its base and altitude.

PROPOSITION IX.

347. Theorem: ***The area of any triangle is equal to half of the product of its base and altitude.***



Statement: Let *1 2 3* be any triangle. The area of *1 2 3* is equal to one-half of the product of *b* and the altitude.

Construction: Construct *4 2 3*, a right-angled triangle, as in the diagram, with the same altitude and base as the altitude and base of the given triangle.

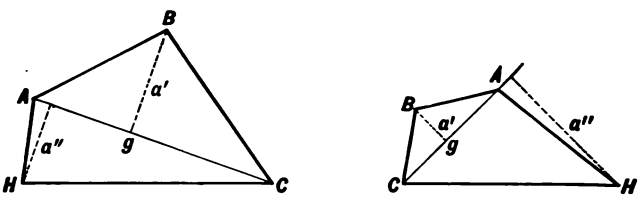
Demonstration: The right-angled triangle *4 2 3* is equivalent to the given triangle *1 2 3*. (BOOK III., PROP. XX., COR. II.) The area of the right-angled triangle is half of the product of *a* and *b* (PROP. VIII.); and, therefore (AXIOM I.), the area of the triangle *1 2 3* is equal to one-half of the product of *a* and *b*.

Conclusion: *1 2 3* being any triangle, it follows, etc.

Corollary. Triangles are proportional in area to the products of their bases and altitudes. If the altitudes are equal, they are proportional to their bases; if their bases are equal, they are proportional to their altitudes.

PROPOSITION X.

348. Theorem: ***A trapezium is equal to half of the product of either of its diagonals and the sum of the perpendiculars from the opposite vertices to the diagonal, or the diagonal produced.***



Statement: Let the diagonal *A C*, or *g*, divide the trapezium *A B C H* into two triangles. The area of the trapezium is then equal to half of the product of *g* by the perpendiculars drawn from *B* and *H* to *g*, or to *g* produced.

Construction: Draw a' and a'' perpendicular to *g* from *B* and *H*.

Demonstration: The trapezium is equivalent to the two triangles. (AXIOM II.) One triangle equals one-half of $a' \times g$, and the other triangle equals one-half of $a'' \times g$. (PROP. IX.) The two triangles, or the trapezium *A B C H*, therefore, equals $\frac{1}{2} g \times (a' + a'')$. (AXIOM III.)

Conclusion: *A B C H* being any trapezium, and *A C* being either of its diagonals, it follows, in general, that: A trapezium is equal to half of the product of either of its diagonals and the sum of the perpendiculars from the opposite vertices to the diagonal, or the diagonal produced.

Scholium. The area of any polygon, regular or irregular, is equal to the sum of the areas of the triangles into which it may be divided.

MISCELLANEOUS EXERCISES.

349. 1. Bisect a given triangle by a line drawn from one of its angles.

2. Find a point within a given triangle from which lines drawn to the several angles will divide the triangle into three equal parts.

3. Trisect a given triangle from a given point within it.

4. The area of an isosceles right-angled triangle is equal to one-fourth the area of the square described upon the base.

5. Construct an isosceles triangle equivalent to a given triangle, its altitude being given.

6. Construct a triangle, the base, an adjacent angle, and the sum of the two other sides being given.

7. Construct a triangle, the base, an adjacent angle, and the difference of the two other sides being given.

8. Construct a square equal to the sum of a given triangle and a given parallelogram.

9. In an equilateral triangle, inscribe three equal circles touching one another, and each touching two sides of the triangle.

10. Prove Prop. IV., by using a line drawn from *B* to *C*.

11. Prove Prop. V., by using a line drawn from *1* to *5*.

12. The maximum polygon, with any number of given sides but unknown base, is that in which the sides are chords, and the unknown base, the diameter of a circle.

SECTION III.—QUADRILATERALS.

PROPOSITION XI.

350. Theorem: ***Parallelograms with equal altitudes are proportional to their bases.***

Statement: Let the parallelograms *A* and *B* be any parallelograms with the same altitude. They are proportional to their bases; that is, $A : B :: CE : FG$.

I. Commensurable bases.

Construction: Let *M* be a parallelogram with the given altitude and a base equal to the common measure of the bases. Lay the common measure of the two bases off on the two bases, as often as it is found in each; as, seven times in *CE*, and nine times in *FG*. From the points of division, draw lines parallel to the sides of the respective parallelograms.

Demonstration: The sixteen parallelograms thus formed are all equal (BOOK III., PROP. XX.);

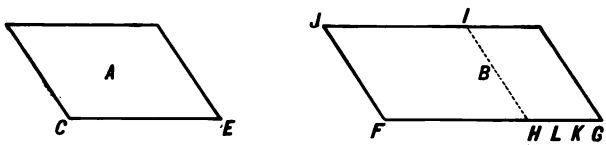
and, $$M = \frac{A}{7} = \frac{B}{9}; \quad \text{or,} \quad A : B :: 7 : 9. \quad \text{(A)}$$

Moreover, $$M = \frac{CE}{7} = \frac{FG}{9}; \quad \text{or,} \quad CE : FG :: 7 : 9. \quad \text{(B)}$$

Then, comparing (A) and (B), (BOOK V., PROP. XI.),

$$A : B :: CE : FG. \qquad Q.\ E.\ D.$$

II. Incommensurable bases.



Construction: Take *F H*, a part of *F G*, equal to *C E*. Draw *H I* parallel to *F J*

Demonstration: The parallelograms *A* and *J H* are equal. (Book III., Prop. XX.) If it is not true that

$$B : JH :: FG : FH, \text{ suppose } B : JH :: FG : FK, \qquad (A)$$

in which *F K* is greater than *F H*. Divide *F G* into parts, each less than *H K*. At least one point of division, as *L*, falls between *H* and *K*. A line may be drawn through *L*, parallel to *F J*, making a parallelogram, *J L*. Since *J L* and *B* have commensurable bases, $B : JL :: FG : FL$. (B)

Comparing the last two proportions,

$$JH : JL :: FK : FL. \text{ (Book V., Prop. XII.)}$$

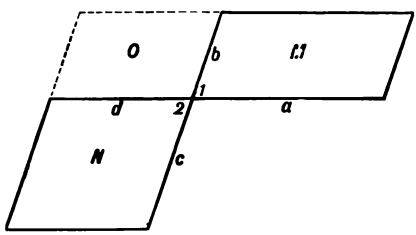
In this proportion, *J H* is less than *J L*, and *F K* should be less than *F L*; but *F K* is greater than *F L*. The supposition that leads to this false conclusion is to be abandoned. The fourth term of the first proportion is, therefore, not greater than *F H*. In similar manner, it may be proved that the fourth term is not less than *F H*. The fourth term is, therefore, *F H*; and it is true that

$B : JH :: FG : FH$; or, by substitution, $B : A :: FG : CE$; or (Book V., Prop. III.), $A : B :: CE : FG$ is true. *Q. E. D.*

Conclusion: As *A* and *B* are any two parallelograms, etc.

PROPOSITION XII.

351. Theorem: ***Parallelograms with an angle in each equal are proportional to the products of the including sides.***



Statement: Let the parallelograms *M* and *N* have the angles *1* and *2* equal. Prove that $M : N :: a \times b : c \times d$.

Construction: Place the parallelogram so that the angles *1* and *2* shall be vertical angles. Produce the sides that are parallel to *a* and *c*, and thus form the figure *O*.

Demonstration: By construction, *O* is a parallelogram. The parallelograms *M* and *O* being between the same parallels and having the same altitude,

$$M : O :: a : d. \qquad \text{(PROP. XI.)}$$

The parallelograms *N* and *O* being between the same parallels and having the same altitude,

$$O : N :: b : c. \qquad \text{(PROP. XI.)}$$

Multiplying the two proportions together, term by term,

$$M \times O : N \times O :: a \times b : c \times d. \quad \text{(BOOK V., PROP. XIV.)}$$

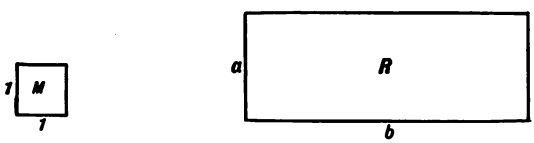
Dividing both terms of the first couplet by *O* gives

$$M : N :: a \times b : c \times d. \quad \text{(BOOK V., PROP. XV., COR.)}$$

Conclusion: Since *M* and *N* are any two, etc.

PROPOSITION XIII.

352. Theorem: ***The area of a rectangle is equal to the product of its base and altitude.***



Statement: Let R be any rectangle; b, its base; and a, its altitude. The rectangle R is equal to the product of a and b, its base and altitude.

Construction: Construct, on any linear unit, as one inch, one foot, or one yard, a square, M, the corresponding unit of area.

Demonstration: R and M being rectangles and, therefore, parallelograms, with an angle in each equal, they are proportional to the products of the two sides a and b, the altitude and the base; that is,

$$R : M :: a \times b : 1 \times 1 \qquad \text{(PROP. XII.)};$$

but M is, by construction, the unit of area, or 1; and, therefore, by substitution,

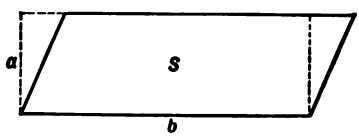
$$R : 1 :: ab : 1; \text{ or, } R = ab. \qquad \text{(BOOK V., PROP. I.)}$$

That is, the rectangle equals the product of its base and altitude.

Conclusion: R being any rectangle, it follows, in general, that: The area of a rectangle is equal to the product of its base and altitude.

PROPOSITION XIV.

353. Theorem: ***The area of a parallelogram is equal to the product of the base and altitude.***



Statement: Let *S* be any parallelogram. **The area of *S* is** equal to the product of its base and altitude.

Construction: On the base of the parallelogram, construct a rectangle having the same altitude as the parallelogram.

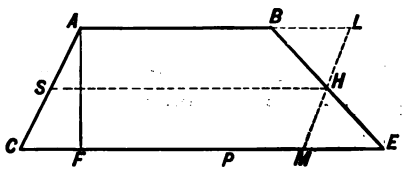
Demonstration: The parallelogram and the rectangle, having the same base and altitude, are equal in area. (BOOK III., PROP. XX.) The area of the rectangle is equal to the product of its base and altitude, *a b*. (PROP. XIII.) The area of the parallelogram is, therefore, equal to the product of *a* and *b* (AXIOM I.); but *a* and *b* are, also, by construction, the altitude and base of the parallelogram. The area of *S* is, therefore, equal to the product of its base and altitude, *a b*.

Conclusion: *S* being any parallelogram, it follows, in general, that: The area of a parallelogram is equal to the product of the base and altitude.

Corollary I. The area of a triangle is equal to half of the *product of its* base and altitude.

PROPOSITION XV.

354. Theorem: ***The area of a trapezoid is equal to the product of its altitude by half of the sum of its parallel sides.***



Statement: Let *A B E C* be any trapezoid. The area is equal to the product of *A F* by half of the sum of *A B* and *C E*.

Construction: Draw *S H*, the diameter, joining the sides not parallel. Draw *L H M* parallel to *A C*. Extend *A B* to *L*.

Demonstration: The trapezoid *A B E C* is equal to the parallelogram *A L M C*, the triangles *B H L* and *M H E* being equal. (BOOK III., PROP. XIX.) The area of the parallelogram *A L M C*, however, and, therefore, the area of the trapezoid *A B E C*, is equal to the product of *A F*, the altitude, by *C M*, the base of the parallelogram. (PROP. XIV.) Since *M E* is equal to *B L*, the line *C M* is, obviously, equal to half of the sum of *A B* and *C E* (BOOK III., PROP. I.); and, the trapezoid is proved equal to the product of its altitude by half the sum of its parallel sides.

Conclusion: As *A B E C* is any trapezoid, it follows, in general, that: The area of a trapezoid is equal to the product of its altitude by half of the sum of its parallel sides.

MISCELLANEOUS EXERCISES.

355. 1. Divide a parallelogram into two parts which shall have a given ratio, by a line drawn parallel to a given line.

2. Bisect a trapezium by a line drawn from one of its angles.

3. Bisect a trapezium by a line drawn from a given point in one of its sides.

4. The triangle contained by either of the sides joining the parallel sides of a trapezoid and the two straight lines drawn from its extremities to the bisection of the opposite side, is half the trapezoid.

5. Divide a given trapezoid in a given ratio by a line drawn through a given point, and terminated by the two parallel sides.

6. If a rectangle be inscribed in a right-angled triangle, and they have the right angle common, the rectangle contained by the segments of the hypotenuse is equal to the sum of the rectangles contained by the segments of the sides about the right angle.

7. The area of a rhombus is equal to half the product of its diagonals.

8. The parallelogram formed by joining the middle points of the adjacent sides of a quadrilateral is one-half of the quadrilateral.

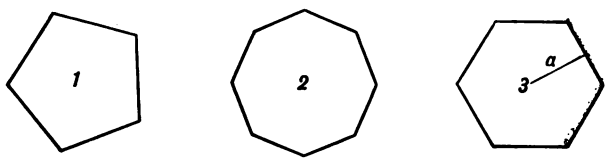
9. Construct a triangle equivalent to a given triangle, two of its sides being given.

10. Construct a triangle equivalent to a given square.

11. Construct a triangle equivalent to a given square, its base and the length of the line from the opposite vertex to the middle point of the base being given.

12. Construct a triangle equivalent to a given polygon.

SECTION IV.—REGULAR POLYGONS.



DEFINITIONS.

356. A **polygon** is a plane figure bounded by straight lines.

357. A **pentagon** is a polygon with five sides ; as *1*.

358. A **hexagon** is a polygon with six sides ; as the figure *3*.

359. A **heptagon** is a polygon with seven sides.

360. An **octagon** is a polygon with eight sides ; as *2*.

361. An **enneagon** is a polygon with nine sides.

362. A **decagon** is a polygon with ten sides.

363. A **hendecagon** is a polygon with eleven sides.

364. A **dodecagon** is a polygon with twelve sides.

365. A **pentedecagon** is a polygon with fifteen sides.

366. A **regular polygon** is an equilateral and equiangular polygon ; that is, all of its sides and angles are equal.

The figure *3* is a *regular hexagon ;* the figure *2*, a *regular octagon.*

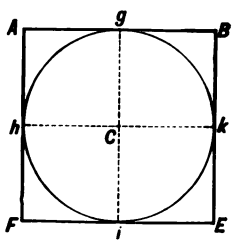
367. A polygon is *inscribed* in a circle when all of its sides are chords ; it is *circumscribed*, when all of its sides are tangents.

368. An **apothem** is a straight line drawn from the center perpendicular to any side of a regular polygon ; as the line *a*.

369. The **perimeter** of a polygon is the broken line that bounds the polygon.

PROPOSITION XVI.

370. Problem: ***To find the side and the area of a circumscribed square.***



Statement: Let the circle whose center is *C* be any circle, and let *A B E F* be any circumscribed square. It is required to find the length of one side and the area of *A B E F*.

Construction: Draw radii to the points of contact.

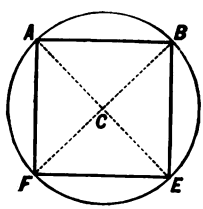
Demonstration: The angles at *g, h, i, k* are all right angles. (BOOK IV., PROX. XIII.) The angles at *A, B, E,* and *F* are, by construction, right angles. The figures *A C, B C, E C,* and *F C* are, therefore, rectangles (BOOK III., PROP. VI.); and, as each has two adjacent sides equal, being equal radii, they are equal squares. (BOOK III., PROP. IV.)

The side of the circumscribed square is, then, obviously equal to the diameter; and the area, to the square of the diameter, or four times the square of the radius.

Conclusion: As the circle whose center is *C* is any circle, it follows, in general, that: The side of a circumscribed square is equal to the diameter, and the area is equal to the square of the diameter, or four times the square of the radius.

PROPOSITION XVII.

371. Theorem: ***To find the side and the area of an inscribed square.***



Statement: Let the circle whose center is C be any circle, and let $ABEF$ be any inscribed square. Find the length of one side of the square AE, and find the area.

Construction: Draw the radii to the vertices of the square.

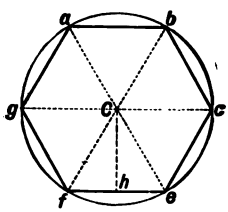
Demonstration: The four sides of the square are four equal chords. The four arcs, AB, BE, EF, and FA, are, therefore, equal (BOOK IV., PROP. III.), and each is a quadrant. The four angles at the center are, therefore, right angles. (PROP. I., COR. III.) The four figures into which the given square is divided are, then, equal right-angled triangles.

The area of any one of these triangles is equal to half of the product of its base, a radius, by its altitude, a radius; that is, $\frac{1}{2} r^2$. (PROP. VIII.) The area of the four triangles, or of the inscribed square, is four times as much; that is, $4 \times \frac{1}{2} r^2$, or $2 r^2$, in which r represents the radius. If the radius is equal to 1, the inscribed square is equal to 2; and the side of the square is equal to $\sqrt{2}$.

Conclusion: As the circle whose center is C is, etc.

PROPOSITION XVIII.

372. Problem: ***To find the side and the area of a regular inscribed hexagon.***



Statement: Let the circle whose center is *C* be any circle, and let *a b c e f g* be any regular inscribed hexagon. It is required to find the area and the length of one side of the inscribed hexagon.

Construction: Draw the radii to the vertices of the hexagon, and draw *C h* from the center perpendicular to *f e*, any side of the hexagon.

Demonstration: The sides *a b*, *b c*, etc., being equal, the arcs *a b*, *b c*, etc., are equal (BOOK IV., PROP. III.); and the angles at the center are, also, all equal. (BOOK IV., PROP. XVI.) Each of the angles at the center is, then, one-sixth of four right angles, or two-sixths, that is, one-third, of two right angles.

The triangles *a C b*, *b C c*, etc., are all equal (BOOK II., PROP. I.), two of the sides in each triangle being radii, and the included angles at the center being each equal to one-third of two right angles.

As the three angles of a triangle equal two right angles (BOOK II., PROP. XVII.), the two other angles in any of the triangles about the center, C, equal two-thirds of two right angles; but the two other angles are equal to each other (BOOK II., PROP. VI.), being opposite equal sides. Each of these angles is, then, equal to one-third of two right angles, and the triangle is equiangular. As the triangles are equiangular, they are, also, equilateral. (BOOK II., PROP. IX.) The sides are, then, all equal, and each is equal to the radius of the circle.

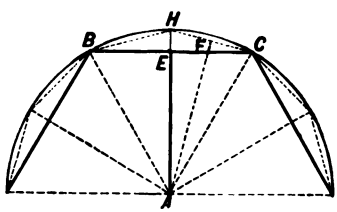
The area of the hexagon is six times the area of any one of its triangles. The area of any one of the triangles is equal to $\frac{1}{2} r \times C h$, half the product of r, its base, by $C h$, its altitude. $C h$ being one side of a right-angled triangle, its square is equal to $C e^2 - e h^2$; but $C e$ is the radius, and $e h$ is equal to half of $e f$ (BOOK IV., PROP. XI.), which is equal to the radius. $C h^2$ is, then, equal to $r^2 - (\frac{1}{2} r)^2 = r^2 - \frac{1}{4} r^2 = \frac{3}{4} r^2$. If $C h^2 = \frac{3}{4} r^2$, $C h = \sqrt{\frac{3}{4} r^2} = \frac{1}{2} r \sqrt{3}$. The area of the triangle is, then, equal to $\frac{1}{2} r \times C h$; or, $\frac{1}{2} r \times \frac{1}{2} r \sqrt{3} = \frac{1}{4} r^2 \sqrt{3}$. The area of the hexagon is $6 \times \frac{1}{4} r^2 \sqrt{3} = \frac{3}{2} r^2 \sqrt{3}$; and, if the radius is 1, the area equals $\frac{3}{2} \sqrt{3}$.

Conclusion: As the circle whose center is C is any circle, and as $a b c e f g$ is any inscribed hexagon, it follows, in general, that: The side of a regular inscribed hexagon is equal to the radius of the circle; and the area is equal to $\frac{3}{2} r^2 \sqrt{3}$.

Scholium. A regular inscribed hexagon may be constructed by using the radius six times as a chord.

PROPOSITION XIX.

373. Problem: ***The perimeter and apothem of a regular inscribed polygon of any number of sides being given, to find the perimeter and apothem of a regular inscribed polygon of double the number of sides.***



Statement: Let *B C* be the side of a given regular inscribed polygon, and let *A H* be drawn perpendicular to *B C*. *A E* is the apothem. With *B C* and *A E* known, it is required to find a side and the apothem of the regular inscribed polygon of double the number of sides.

Construction: Draw *B H* and *H C*, and draw *A F* perpendicular to *H C*; also, draw radii perpendicular to the sides of the given polygon.

Demonstration: The arcs *B H*, *H C*, etc., are all equal. (BOOK IV., PROP. XI.) The chords *B H*, *H C*, etc., are, then, all equal (BOOK IV., PROP. II); and the angles made by these chords are equal. (PROP. II.)

The figure thus drawn, *B H C*, etc., is, then, a regular polygon of double the number of sides of the given polygon. As *H C* is a chord thus constructed, it is a side of the required polygon, and *A F* is its apothem.

Let S represent a side of the given polygon ; A, its apothem ; s, a side of the required polygon, and a, its apothem.

Then, $s^2 = HE^2 + EC^2$ (BOOK III., PROP. XXIV.) ;

and, $HE^2 = (R - A)^2$, since, $HE = AH - AE = R - A$;

also, $EC^2 = AC^2 - AE^2 = R^2 - A^2$.

By substitution, $s^2 = (R - A)^2 + (R^2 - A^2)$

$= R^2 - 2AR + A^2 + R^2 - A^2$

$= 2R^2 - 2AR = 2(R^2 - AR)$; and,

$s = \sqrt{2(R^2 - AR)}$, and, if $R = 1$, $s = \sqrt{2 - 2A}$. *Q. E. F.*

Now, $AF^2 = AH^2 - HF^2$ (BOOK III., PROP. XXIV.);

that is, $a^2 = R^2 - \left(\frac{s}{2}\right)^2 = R^2 - \frac{s^2}{4} = \frac{4R^2 - s^2}{4} = \frac{1}{4}(4R^2 - s^2)$.

If $a^2 = \frac{1}{4}(4R^2 - s^2)$, $a = \frac{1}{2}\sqrt{4R^2 - s^2}$. *Q. E. F.*

When $R = 1$, $a = \frac{1}{2}\sqrt{4 - s^2}$.

Conclusion: Since BC is a side of any regular inscribed hexagon, it follows, in general, that: The side of a regular inscribed polygon of double the number of sides is equal to $\sqrt{2(R^2 - AR)}$, or, $\sqrt{2 - 2A}$, when $R = 1$, and the apothem equals $\frac{1}{2}\sqrt{4R^2 - s^2}$; or, $\frac{1}{2}\sqrt{4 - s^2}$, when $R = 1$.

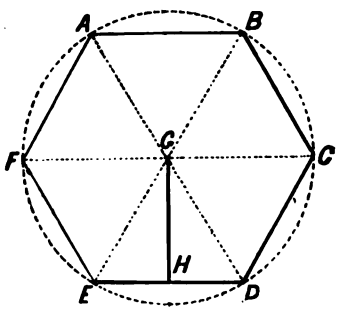
Corollary I. The area of a regular inscribed polygon of double the number of sides is equal to

$$\frac{n}{4}\sqrt{2(R^2 - AR)(4R^2 - s^2)}.$$

Scholium. If a regular inscribed polygon of any number of sides is given, a regular inscribed polygon of double the number of sides may be constructed by drawing radii perpendicular to the sides and the chords of the halves of the arcs thus constructed.

PROPOSITION XX.

374. Theorem: ***The area of any regular polygon is equal to one-half of the product of its apothem and perimeter.***



Statement: Let *A B C D E F* be any regular polygon, and *G H*, its apothem. The area of the polygon is equal to one-half of the product of *G H* and the perimeter, *A B C D E F*.

Construction: Circumscribe the circumference of a circle about the polygon, and draw the radii *G E*, *G D*, etc.

Demonstration: The polygon is thus divided into as many triangles as it has sides. The triangles are all equal. (BOOK II., PROP. III.) The apothem is a constant quantity, being the perpendicular distance from the center to equal chords. (BOOK IV., PROP. IV.)

The area of each triangle is one-half the product of the apothem by its base. (PROP. IX.) The area of all the triangles, or of the regular polygon, is one-half of the product of the apothem by the sum of the bases, or the perimeter.

Conclusion: *A B C D E F* being any regular polygon, etc.

MISCELLANEOUS EXERCISES.

375. 1. The interior angles of a polygon equal two right angles taken as many times as there are sides less two.

2. The exterior angles of any polygon, made by producing the sides in the same order, equal four right angles.

3. The regular inscribed hexagon is double the inscribed, and one-half of the circumscribed equilateral triangle.

4. The area of the regular inscribed dodecagon is equal to three times the square of the radius.

5. In a given square, inscribe four equal circles, tangent each to two of the others and to one side of the square.

6. The figure bounded by the five diagonals of a regular pentagon is itself a regular pentagon.

7. The regular inscribed hexagon is three-fourths of the regular circumscribed hexagon.

8. The square of the side of the regular inscribed decagon, together with the square of the radius, is equal to the square of the side of the regular inscribed pentagon.

9. If two circumferences of circles intersect, the maximum straight line through a point of section, and terminated by the circumferences, is parallel to the line joining the centers.

10. Tangents from the same point to a circle are equal.

11. The sums of the opposite sides of a quadrilateral circumscribed about a circle are equal.

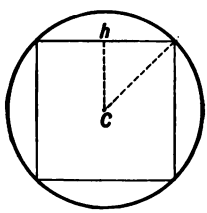
12. Construct circles on the sides of any quadrilateral as diameters, and prove that the common chord of any two consecutive circles is parallel to that of the other two circles.

13. Through any point, draw a secant to a circle so that the intercepted chord may equal a given straight line.

SECTION V.—CIRCLES.

PROPOSITION XXI.

376. Problem: ***To find the circumference of a circle.***



Statement: Let the circle whose center is *C* represent any circle. It is required to find its circumference.

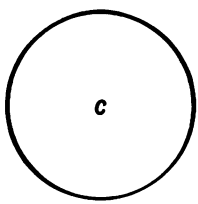
Construction: Construct the inscribed square. Draw *C h*, the apothem of the inscribed square.

Demonstration: As the perimeter of the inscribed square is known (PROP. XVII.), the side and perimeter of the regular inscribed polygons of *8, 16, 32, 64,* etc., sides may be found in succession. (PROP. XIX.) The perimeter of the inscribed polygon of *8,192* sides is thus found to be *3.141592* + (or, *3.1416*) × *2 R*; or, *3.1416* × *D*; or, representing *3.1416* by π, the perimeter equals $2\pi R$, or πD. This, which is correct to three decimal places, represents the length of the circumference of the circle in terms of the radius or diameter. For, as the number of sides of the inscribed polygon is increased, its perimeter approaches its limit, the circumference; and the apothem, its limit, the radius of the circle.

Conclusion: As the circle whose center is *C*, etc.

PROPOSITION XXII.

377. Problem: ***To find the area of a circle.***



Statement: Let the circle whose center is C be any circle.

Construction: Construct a regular inscribed polygon of any number of sides.

Demonstration: By increasing the number of sides, the perimeter finally reaches its limit, the circumference; and the apothem its limit, the radius. The area of the polygon becomes the area of the circle. As the area of a regular inscribed polygon is equal to one-half of the product of its apothem and perimeter (PROP. XX.), the area of the circle equals one-half of the product of its circumference, $2\pi R$ (PROP. XXI.), and radius, R; that is, $\frac{1}{2}$ of $2\pi R^2$, or πR^2.

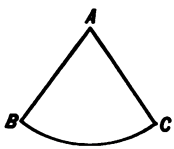
Conclusion: The circle whose center is C being any circle, it follows, in general, that: The area of any circle is πR^2.

Scholium. The circumference equals π times the diameter; and, the area equals π times the square of the radius.

Exercise. 1. Find the area of a circle whose radius is 2.

PROPOSITION XXIII.

378. Problem: ***To find the area of a sector of a circle.***



Statement: Let *A B C* be any sector of any circle. It is required to find its area.

Demonstration: By a course of reasoning similar to that in PROPS. I., VI., and XI., it is demonstrated that sectors of circles are proportional to their arcs, or to their included angles; that is,

Circle : Sector : : 360° : number of degrees in sector.

The area of the sector, then, equals the area of the circle, $\pi R^2 \times$ arc (or angle) of the sector, divided by 360. (BOOK V., PROP. I., COR. I.)

Conclusion: The given sector being any sector, it follows, in general, that: The area of a sector of a circle, etc.

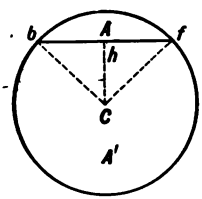
Exercises. 1. Find the locus of the centers of circles passing through two given points.

2. Find the locus of the centers of circles touching a given straight line and having a given radius.

3. Find the locus of the centers of circles touching two given straight lines.

PROPOSITION XXIV.

379. Problem: *To find the area of a segment of a circle.*



Statement: Let the circle whose center is C be any circle. Let A be any segment of a circle. Find its area.

Construction: Draw $C\,b$ and $C\,f$, and $C\,h$ perpendicular to $b\,f$.

Demonstration: The segment is equivalent to the sector less the triangle $C\,b\,f$.

The area of the sector equals $\frac{\pi R^2 \times \text{arc } b\,f}{360}$. (PROP. XXIII.)

The area of the triangle equals

$$\tfrac{1}{2}\,b\,f \times C\,h = \tfrac{1}{2}\,b\,f \times \sqrt{R^2 - \frac{b\,f^2}{4}}.$$

The area of the segment equals

$$\frac{\pi R^2 \times \text{arc } b\,f}{360} - \tfrac{1}{2}\text{ chord } b\,f\sqrt{R^2 - \frac{b\,f^2}{4}}.$$

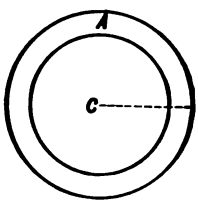
The area of the segment greater than a semicircle is equal to the re-entrant sector A' + the triangle.

Conclusion: As the circle whose center is C, etc.

Corollary I. The area of a part of a circle between two chords is the difference of two segments.

PROPOSITION XXV.

380. Problem: ***To find the area of a circular ring.***



Statement: Let the circles whose centers are at C be any circles forming A, a circular ring, between their circumferences. It is required to find the area of A.

Construction: Draw the radii, R being that of the larger circle, and r, the radius of the smaller circle.

Demonstration: The area of the ring is evidently the difference between the areas of the two circles. The smaller circle equals πr^2. (PROP. XXII.) The larger circle equals πR^2. (PROP. XXII.) The ring,

$$A = \pi R^2 - \pi r^2 = \pi (R^2 - r^2).$$

Conclusion: As the circles whose centers are at C, etc.

MISCELLANEOUS EXERCISES.

381. 1. The area of any circle is equal to four times the area of the circle described upon its radius as a diameter.

2. In concentric circles, chords to the outer circle that are tangent to the inner circle are bisected at the point of contact.

3. The middle points of the sides of any triangle are concyclic with the foot of any perpendicular from any vertex to the opposite side.

4. The middle points of the sides of any triangle and the feet of the perpendiculars to the sides from the opposite vertices are concyclic.

5. In a given circle, inscribe three equal circles touching one another and the given circle.

6. Construct a segment of a circle on a given line so as to contain an inscribed angle equal to a given angle.

Construct an angle equal to the given angle, having the given line for one side and an indefinite line for the other side. The latter line being a tangent and the former a chord, find the center.

7. Find the locus of the vertices of triangles having a given base and a given vertical angle.

8. Construct triangles, having given the base, the vertical angle, and (*a*) the altitude; (*b*) the median to the base; (*c*) either side; (*d*) the point in the base at which the base is cut by the perpendicular from the vertical angle; or, (*e*) by the bisectrix of the vertical angle; (*f*) the sum of the two sides; (*g*) the difference of the two sides.

9. If the circumferences of two circles intersect each other orthogonally, the tangent to either circle at the point of intersection will pass through the center of the other circle.

Two circumferences of circles intersect orthogonally when their tangents at the point of intersection are perpendicular to each other.

10. The square of the distance between the centers of two circles intersecting each other orthogonally is equal to the sum of the squares of the radii.

11. The angles of an orthocenter, or pedal triangle of any acute-angled triangle are bisected by the perpendiculars from the opposite vertices of the acute-angled triangle.

1. Find three sets of concyclic points. The parts of any angle of the pedal triangle are then easily proved equal by showing that they are equal to angles which are complements of the opposite angle of the original triangle.

12. Find the locus of the orthocenter of any triangle, the base and vertical angle being given.

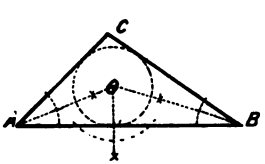
Analyze, by using the principle of concyclic points, noting the value of the angle opposite the given vertical angle.

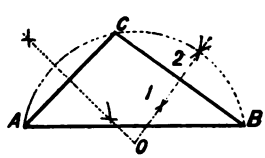
13. Find the locus of the bisectrices of the angles of any triangle, the base and opposite angle being given.

Analyze, drawing all lines mentioned and noting value of angles opposite the base.

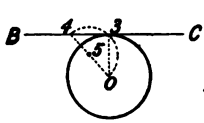
CONSTRUCTIONS.

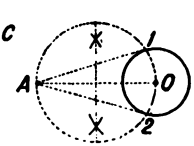
382. Prove the following constructions:





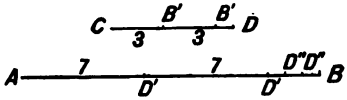
1. *In a given triangle, inscribe a circle.*
2. *About a given triangle, circumscribe a circle.*





3. *A tangent to a given circle from different points.*

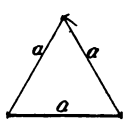
4. Find the ratio of two commensurable straight lines,

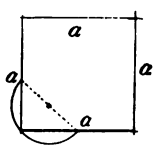


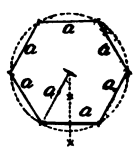
As in finding G. C. D., use each remainder as the divisor and the preceding divisor as a dividend; thus, (1) CD is found twice in AB, with $D'B$ as a remainder; (2) $D'B$ is found twice in CD, with $B'D$ as a remainder; (3) $B'D$ is found three times in $D'B$ without a remainder. $B'D$ is, then, the common measure, being found seven times in CD and seventeen times in AB. The ratio $CD : AB = 7 : 17$; or, $AB : CD = 17 : 7$.

5. Find the ratio of two commensurable arcs.

6. Equilateral triangle, one side given.

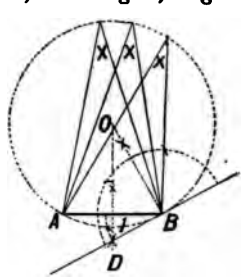






7. Square, a side given. 8. Regular hexagon, a side given.

9. Inscribed triangle, square, or hexagon, in given circle,



10. On a given straight line, a segment of a circle, to contain a given angle.

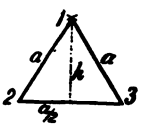
Construct at B an angle equal to the angle *1*. AB is a chord; DB, a tangent. These give the center, *O*, by intersection of loci. *X* is the required angle.

NUMERICAL EXERCISES.

383. 1. Find the area of an equilateral triangle, in terms of a side.

1. Area $= h \times \frac{a}{2}$; but, $h^2 = a^2 - \frac{a^2}{4} = \frac{3}{4} a^2$; $\therefore$

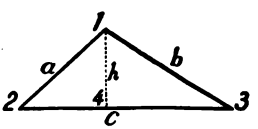
$$h = \frac{a}{2}\sqrt{3}; \text{ and, area} = \frac{a}{2} \times \frac{a}{2}\sqrt{3} = \frac{a^2}{4}\sqrt{3}.$$



2. Find the altitude of any triangle, in terms of the sides.

$h^2 = a^2 - 2\,4^2$; $b^2 = a^2 + c^2 - 2c \times 2\,4$.

$\therefore 2\,4 = \frac{a^2 + c^2 - b^2}{2c}$; and, $h^2 = a^2 - \left(\frac{a^2 + c^2 - b^2}{2c}\right)^2 = \frac{4a^2c^2 - (a^2 + c^2 - b^2)^2}{4c^2}$.



$$\therefore h^2 = \frac{(2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2)}{4c^2}$$

$$= \frac{\{(a + c)^2 - b^2\}\{b^2 - (a - c)^2\}}{4c^2}.$$

$$\therefore h = \frac{(a + c + b)(a + c - b)(b + a - c)(b - a + c)}{4c^2}.$$

Let $2s = a + b + c$; then, $a + c - b = 2(s - b)$; and,

$b + a - c = 2(s - c)$; and, $b - a + c = 2(s - a)$.

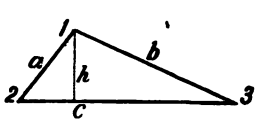
$$\therefore h^2 = \frac{2s \times 2(s - a) \times 2(s - b) \times 2(s - c)}{4c^2}.$$

$$\therefore h = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}.$$

3. Find the area of any triangle in terms of the sides.

$$\text{Area} = \frac{c}{2} \times h$$

$$= \frac{c}{2} \times \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}.$$



$\therefore$ Area $= \sqrt{s(s - a)(s - b)(s - c)}$; **$s$ being half the perimeter.**

BOOK VII.—SIMILAR FIGURES.

SYNOPSIS.

SECTION I.—TRIANGLES.

384. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. I. Triangles that are mutually equiangular are similar.

PROP. II. Triangles with proportional sides are similar.

PROP. III. Triangles with an angle in each equal, and the including sides proportional, are similar.

PROP. IV. Triangles with sides respectively parallel are similar.

PROP. V. Triangles with sides respectively perpendicular are similar.

3. MISCELLANEOUS EXERCISES.

SECTION II.—PARTS OF TRIANGLES.

385. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. VI. Lines drawn parallel to the base of a triangle divide the sides proportionally.

PROP. VII. Lines drawn from the vertex of a triangle to the base divide the base and lines drawn parallel to it into proportional parts.

PROP. VIII. Lines that divide the sides of a triangle are parallel to the base.

PROP. IX. Either side of a right-angled triangle is a mean proportional between the hypotenuse and its projection upon the hypotenuse made by the perpendicular drawn from the right angle to the hypotenuse.

PROP. X. In any right-angled triangle, the perpendicular from the right angle to the hypotenuse is a mean proportional between the segments of the hypotenuse.

3. MISCELLANEOUS EXERCISES.

SECTION III.—POLYGONS.

386. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XI. Regular polygons of the same number of sides are similar.

PROP. XII. Similar polygons may be divided into the same number of similar triangles, similarly placed.

PROP. XIII. Polygons composed of the same number of triangles, similar each to each, and similarly placed, are similar.

PROP. XIV. The perimeters of similar polygons are proportional to any homologous lines.

PROP. XV. The perimeters of similar regular polygons are proportional to the apothems, the radii, or the diameters of the inscribed, or of the circumscribed polygons.

3. MISCELLANEOUS EXERCISES.

SECTION IV.—CIRCLES.

387. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XVI. The segments of intersecting chords in a circle are reciprocally proportional.

PROP. XVII. Secants drawn from the same point without a circle are reciprocally proportional to their external segments.

PROP. XVIII. If, from any point without a circle, a tangent and secant are drawn, the tangent is a mean proportional between the secant and its external segment.

PROP. XIX. Circumferences of circles are proportional to the diameters or to their radii.

PROP. XX. Arcs of similar sectors of circles are proportional to their radii or to the diameters of the circles; or, to the chords of the arcs.

3. MISCELLANEOUS EXERCISES.

SECTION V.—PROPORTIONAL AREAS.

388. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. XXI. Similar triangles are proportional to the squares of their homologous sides.

PROP. XXII. Similar polygons are proportional to the squares of any homologous lines.

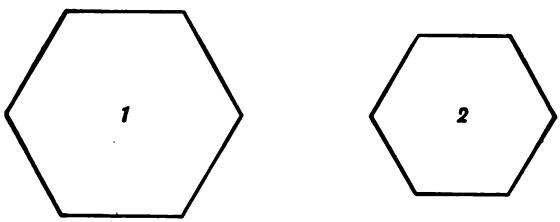
PROP. XXIII. Three similar figures having been constructed on the hypotenuse and the sides of a right-angled triangle, the figure constructed on the hypotenuse is equal to the sum of the two other figures.

PROP. XXIV. The areas of circles are proportional to the squares of their radii.

PROP. XXV. The areas of similar sectors of a circle are proportional to the squares of the radii, the chords, or any homologous lines.

3. MISCELLANEOUS EXERCISES.

SECTION I.—TRIANGLES.

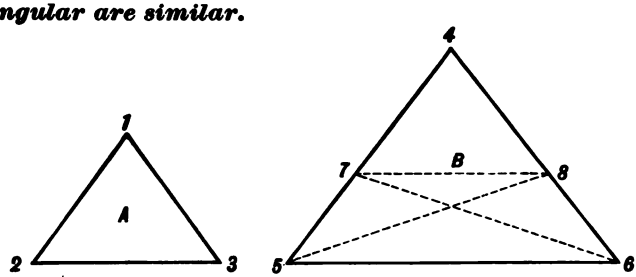


DEFINITIONS.

389. Similar figures are figures having the angles taken in the same order equal, and the sides proportional; as *1* and *2*.

PROPOSITION I.

390. Theorem: ***Triangles that are mutually equiangular are similar.***



Statement: Let the triangles *A* and *B* have the angles *1*, *2*, and *3* respectively equal to the angles *4*, *5*, and *6*. Prove

$$1\,2 : 4\,5 :: 1\,3 : 4\,6 :: 2\,3 : 5\,6.$$

Construction: Take *4 7*, a part of *4 5*, equal to *1 2*; and *4 8*, a part of *4 6*, equal to *1 3*. Draw *7 8*, *7 6*, and *8 5*.

Demonstration: The triangles *1 2 3* and *4 7 8* are equal. (BOOK II., PROP. I.) The angles *4 7 8* and *4 5 6* are, then, equal, as each is equal to the angle *2*. The lines *7 8* and *5 6* are, then, parallel (BOOK I., PROP. XVIII., COR. I.); and the triangles *7 8 5* and *7 8 6* are equal. (BOOK III., PROP. XX., COR. II.)

Then, $4\,7\,8 : 7\,8\,5 :: 4\,7 : 7\,5$ (BOOK VI., PROP. VI.);
and, also, $4\,7\,8 : 7\,8\,6 :: 4\,8 : 8\,6$. The second terms being equal, $4\,7 : 7\,5 :: 4\,8 : 8\,6$ (BOOK V., PROP. XI.);
by composition, $4\,7 : 4\,5 :: 4\,8 : 4\,6$. (BOOK V., PROP. IV.)
Substituting for *4 7* and *4 8* their equals, *1 2* and *1 3*, gives,

$$1\,2 : 4\,5 :: 1\,3 : 4\,6.$$

In similar manner, $1\,2 : 4\,5 :: 2\,3 : 5\,6$.

As the triangles have their angles equal, by hypothesis, and their sides proportional, they are similar.

Conclusion: As the triangles *1 2 3* and *4 5 6* are, etc.

Corollary I. Triangles with two angles of the one equal, each to each, to two angles of the other, are similar.

Corollary II. Isosceles triangles with the vertical angles equal are similar.

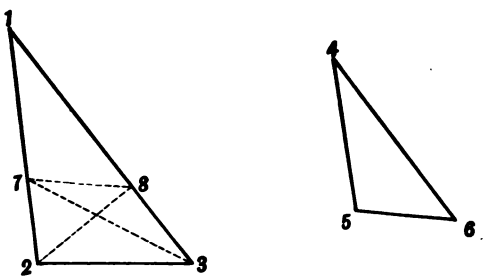
Exercises. 1. The medians of a triangle are concurrent.

2. The medians of a triangle intersect at a point of trisection, the greater segment in each being toward the angular point.

3. Find the locus of the centroid, or intersection of the medians, of a triangle, the base and the vertical angle being given.

PROPOSITION II.

391. Theorem: ***Triangles with proportional sides are similar.***



Statement: Let *1 2 3* and *4 5 6* be triangles whose sides, taken in the same order, are proportional. Prove the triangles are mutually equiangular and, therefore, similar.

Construction: Take *1 7*, a part of *1 2*, equal to *4 5*; and, *1 8*, a part of *1 3*, equal to *4 6*. Draw *7 8*, *7 3*, and *8 2*.

Demonstration: As the triangles *1 7 8* and *7 8 2* have the same altitude,

$$1\,7\,8 : 7\,8\,2 :: 1\,7 : 7\,2$$ (BOOK VI., PROP. VI.);

and, also, $$1\,7\,8 : 3\,7\,8 :: 1\,8 : 8\,3.$$ (BOOK VI., PROP. VI.)

By hypothesis, as the sides are proportional,

$$1\,2 : 4\,5 :: 1\,3 : 4\,6;$$

or, by substitution, $$1\,2 : 1\,7 :: 1\,3 : 1\,8,$$

and, by division, $$1\,7 : 7\,2 :: 1\,8 : 8\,3.$$ (BOOK V., PROP. V.)

The first couplets, therefore, form a proportion (BOOK V., PROP. XI.); that is, $$1\,7\,8 : 7\,8\,2 :: 1\,7\,8 : 3\,7\,8.$$

In this proportion, the first and third terms are the same; the second and fourth terms are, therefore, equal. That is, the triangles *7 8 2* and *3 7 8* are equal. They have, however, the same base, *7 8;* and they must have, therefore, the same altitude. (BOOK VI., PROP. VI., COR. I.) The line *2 3*, containing the vertices of the triangles, is, therefore, parallel to *7 8*. (BOOK I., PROP. XIX., COR. I.)

If *7 8* and *2 3* are parallel, the angles *1 7 8* and *1 2 3* are equal; and, also, the angles *1 8 7* and *1 3 2*. (BOOK I., PROP. XVII.) The angle at *1* is common. The two triangles *1 7 8* and *1 2 3* are, then, mutually equiangular and, therefore, similar. (PROP. I.) Then, the sides are proportional; as,

$$1\,2 : 1\,7 :: 2\,3 : 7\,8;$$

but, by hypothesis, $1\,2 : 4\,5 :: 2\,3 : 5\,6$. The antecedents in the two proportions being the same, the consequents form a proportion (BOOK V., PROP. XII.), that is,

$1\,7 : 4\,5 :: 7\,8 : 5\,6$. In this proportion, as *1 7* equals *4 5*, by construction, *7 8* and *5 6* are equal.

The triangles *1 7 8* and *4 5 6* have, then, three sides of the one equal to three sides of the other, each to each, and they are, therefore, equal in all their parts. (BOOK II., PROP. III.)

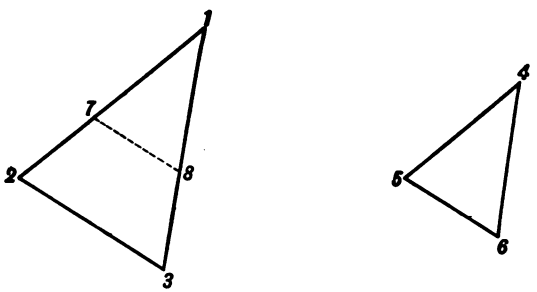
The triangle *1 7 8* has, however, been proved mutually equiangular with the triangle *1 2 3*. The triangles *4 5 6* and *1 2 3* are, therefore, mutually equiangular and similar. (PROP. I.)

Conclusion: The triangles *1 2 3* and *4 5 6* being, etc.

Corollary I. A line that divides the sides of a triangle into proportional parts is parallel to the base.

PROPOSITION III.

392. Theorem: ***Triangles with an angle in each equal, and the including sides proportional, are similar.***



Statement: Let the triangles *1 2 3* and *4 5 6* have the angles *1* and *4* equal, and the included sides proportional; that is, *1 2 : 4 5 : : 1 3 : 4 6.* The triangles are similar.

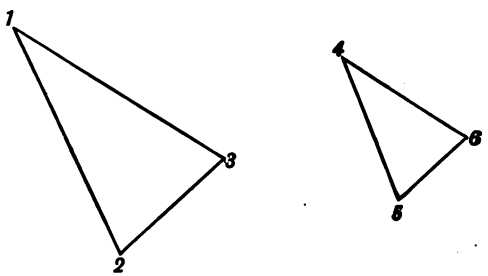
Construction: Take *1 7*, a part of *1 2*, equal to *4 5*, and take *1 8*, a part of *1 3*, equal to *4 6*. Draw *7 8*.

Demonstration: As *4 5 : 1 2 : : 4 6 : 1 3*, by substituting equals, *1 7 : 1 2 : : 1 8 : 1 3.*
The lines *7 8* and *2 3* are, then, parallel. (PROP. II., COR. I.) The angles *1 7 8* and *1 2 3* are, therefore, equal; and, also, the angles *1 8 7* and *1 3 2*. (BOOK I., PROP. XVII.) The triangles *1 7 8* and *1 2 3* are, then, similar. (PROP. I.) The triangles *1 7 8* and *4 5 6* are, however, equal in all their parts. (BOOK II., PROP. I.) The triangles *4 5 6* and *1 2 3* are, therefore, also, mutually equiangular and similar.

Conclusion: *1 2 3* and *4 5 6* being any two triangles, etc.

PROPOSITION IV.

393. Theorem: ***Triangles with sides respectively parallel are similar.***



Statement: *1 2* and *4 5* are parallel; *1 3* and *4 6*; and, also, *2 3* and *5 6*. Prove the triangles similar.

Demonstration: The angles *1* and *4* are equal (BOOK I., PROP. XX.); also, *2* and *5*; and *3* and *6*. The triangles, being mutually equiangular, are similar. (PROP. I.)

Conclusion: The triangles *1 2 3* and *4 5 6* being, etc.

Exercises. 1. The maximum polygon, all the sides being given except one, is that which can be inscribed in a semicircle, the unknown side being the diameter.

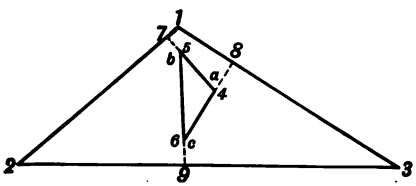
Assume the maximum polygon. Prove the triangle, made by the base, or unknown line, and any two lines drawn to its extremities from any vertex of the polygon, a right-angled triangle.

2. The maximum polygon, all the sides being given, is that which can be inscribed in a circle.

3. The rectangle has less perimeter than any other parallelogram with equal base and altitude.

PROPOSITION V.

394. Theorem: ***Triangles with sides respectively perpendicular are similar.***



Statement: Let *1 2 3* and *4 5 6* be any two triangles with sides respectively perpendicular; as, *4 5* perpendicular to *1 2*, *5 6* perpendicular to *2 3*, and *6 4* perpendicular to *1 3*. The triangles are then similar; that is, the angles included between sides respectively perpendicular are equal; as, angle *1* to angle *4*, angle *2* to angle *5*, and angle *3* to angle *6*; and the sides, taken in the same order, are proportional; as,

$$1\,2 : 4\,5 :: 2\,3 : 5\,6 :: 3\,1 : 6\,4.$$

Construction: Produce the sides of any angle so that they meet the perpendicular sides of the corresponding angle in the other figure; as, *4 5* perpendicular to *1 2* at *7*, and *6 4* perpendicular to *1 3* at *8*.

Demonstration: In the quadrilateral *4 8 1 7*, the angles *8* and *7* being right angles, by construction, the angle *1* is the supplement of the angle *a*. (BOOK III., PROP. I.) The angle *4* is, also, the supplement of the angle *a*. (BOOK I., PROP. VI.) The angles *1* and *4* are, therefore, equal.

In similar manner, the angle *2* is shown to be equal to the angle *5* ; and *3*, to *6*.

The two triangles are, then, mutually equiangular ; and, therefore, similar. (PROP. I.)

Conclusion: The triangles *1 2 3* and *4 5 6* being, etc.

Scholium. The proportional sides in similar triangles are sides taken in the same order, including equal angles ; and the equal angles are the ones included by parallel, perpendicular, or proportional sides.

MISCELLANEOUS EXERCISES.

395. 1. In the figure on the opposite page, indicate three sets of concyclic points, each set containing four points.

2. Inscribe in a circle a triangle similar to a given triangle.

3. Circumscribe about a circle a triangle similar to a given triangle.

4. Construct a triangle similar to a given triangle and having a given perimeter.

5. The distance from the center of a circle to a chord 8 inches long is 4 inches. Find the distance from the center of a chord 5 inches long.

6. Is the triangle whose sides are 3, 4, and 5, a right, acute, or obtuse-angled triangle ? Find its area.

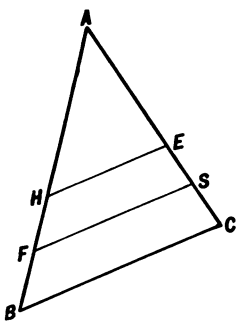
7. What kind of triangle is that whose sides are 7, 7, 9 ? 7, 7, 7 ? 5, 7, 9 ? 10, 15, 20 ? 12, 16, 20 ? Find the areas.

8. The sides of a right-angled triangle are 6 and 10. What is the distance of the vertex of the right angle from the hypotenuse ?

SECTION II.—PARTS OF TRIANGLES.

PROPOSITION VI.

396. Theorem: ***Lines drawn parallel to the base of a triangle divide the sides proportionally.***



Statement: Let *A B C* be any triangle, with *H E* and *F S* drawn parallel to *B C*, the base. The sides *A B* and *A C* are divided proportionally; that is,

$$AH : AE :: HF : ES :: FB : SC.$$

Demonstration: Since the lines *H E* and *F S* are parallel to the base *B C*, the corresponding angles at *H*, *F*, and *B*, and, also, at *E*, *S*, and *C*, are equal. (BOOK I., PROP. XVII.) The triangles *A H E*, *A F S*, and *A B C* are, then, equiangular and similar. (PROP. I.) It follows, that

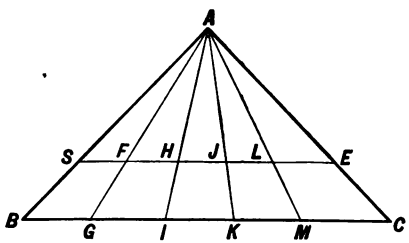
$$AH : AE :: AF : AS :: AB : AC;$$ or,

$$AH : AE :: HF : ES :: FB : SC.$$ (BOOK V., PROP. V.)

Conclusion: Since *A B C* is any triangle, etc.

PROPOSITION VII.

397. Theorem: ***Lines drawn from the vertex of a triangle to the base divide the base and lines parallel to it into proportional segments.***



Statement: Lines drawn from the vertex, *A*, to the base, *B C*, divide *B C*, and *S E*, any line drawn parallel to *B C*, into proportional segments; that is,

$$SF : BG :: FH : GI :: HJ : IK :: JL : KM :: LE : MC.$$

Demonstration: Since *S E* is parallel to *B C*, it follows that the angles at *S*, *F*, *H*, *J*, *L*, and *E* are equal to the corresponding angles at *B*, *G*, *I*, *K*, *M*, and *C*. (BOOK I., PROP. XVII.) The triangles *A S F* and *A B G* are, then, similar (PROP. I.); also, *A F H* and *A G I*; *A H J* and *A I K*; *A J L* and *A K M*; and *A L E* and *A M C*. As the triangles are similar,

$SF : BG :: AF : AG$; and $FH : GI :: AF : AG$; and, therefore, $SF : BG :: FH : GI$. (BOOK V., PROP. XI.)

In similar manner, it may be proved that

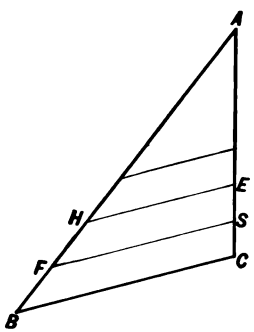
$FH : GI :: HJ : IK$; and $HJ : IK :: JL : KM$; and $JL : KM :: LE : MC$; and, therefore (BOOK V., PROP. XI.),

$$SF : BG :: FH : GI :: HJ : IK :: JL : KM :: LE : MC.$$

Conclusion: *A B C* being any triangle, etc.

PROPOSITION VIII.

398. Theorem: ***Lines that divide the sides of a triangle proportionally are parallel to the base.***



Statement: Let *A B C* be any triangle; *B C*, its base; and *H E* and *F S*, any lines dividing the sides *A B* and *A C* into proportional parts; as,

$AH : AE :: HF : ES :: FB : SC$. Prove that the lines *H E* and *F S* are parallel to the base.

Demonstration: By composition (BOOK V., PROP. IV.),

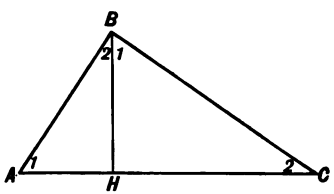
$$AH : AE :: AF : AS :: AB : AC.$$

The triangles *A H E*, *A F S*, and *A B C* have, then, the angle at *A* common, and the including sides proportional. The triangles are, then, similar, and have their corresponding angles equal. (PROP. III.) Since the corresponding angles at *H*, *F*, and *B* are equal, the lines *H E*, *F S*, and *B C* are parallel; that is, *H E* and *F S* are parallel to the base *B C*.

Conclusion: Since *A B C* is any triangle, etc.

PROPOSITION IX.

399. Theorem: ***Either side of a right-angled triangle is a mean proportional between the hypotenuse and its projection upon the hypotenuse.***



Statement: Let *A B C* be any right-angled triangle, right-angled at *B*. Prove that

$$AC : AB :: AB : AH ; \text{ or, } AC : BC :: BC : CH.$$

Demonstration: The two triangles *A B C* and *A B H* are equiangular, as the angles at *1* are common to the two triangles; and the angle at *B*, in *A B C*, is equal to the angle at *H*, in *A B H*, each being a right angle. The triangles are, therefore, similar (PROP. I., COR. I.); and

$$AC : AB :: AB : AH ; \text{ also, } AC : BC :: BC : CH.$$

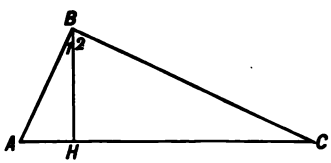
Conclusion: *A B C* being any right-angled triangle, etc.

Corollary I. The square on either side of a right-angled triangle is equal to the rectangle of the hypotenuse and its projection upon the hypotenuse.

Exercise. 1. The sides being *3*, *4*, and *5*, find projection of *3*.

PROPOSITION X.

400. Theorem: ***In any right-angled triangle, the perpendicular from the right angle to the hypotenuse is a mean proportional between the segments of the hypotenuse.***



Statement: Let *A B C* be any right-angled triangle, right-angled at *B*. Let *B H* be the perpendicular from the right angle *B* to the hypotenuse.

Demonstration: The triangles *A B C* and *A B H* are mutually equiangular, as each has a right angle, and one angle, *A*, common to the two triangles. The third angle, *1*, of the smaller triangle is, therefore, equal to *C*, the third angle of the larger triangle. In similar manner, the triangle *B H C* being similar to *A B C*, the angle *2* of the smaller triangle is proved equal to the angle *A* of the larger triangle. The angles at *H* are right angles. The triangles *A B H* and *B H C* are, then, mutually equiangular, each being similar to *A B C*; and they are, then, similar to each other. (PROP. I.) *A H*, of the first triangle, opposite the angle *1*, is to *B H*, of the second triangle, opposite the equal angle *C*, as *B H*, of the first triangle, opposite the angle *A*, is to *H C*, of the second triangle, opposite the equal angle *2*; that is,

$$AH : BH :: BH : HC.$$

Conclusion: *A B C* being any right-angled triangle, etc.

Corollary I. The square on the perpendicular from the vertex of a right-angled triangle to the hypotenuse equals the rectangle of the segments of the hypotenuse.

MISCELLANEOUS EXERCISES.

401. 1. Divide a given straight line into parts proportional to given straight lines.

2. Divide a given straight line into any number of equal parts.

3. Construct a fourth proportional to three given straight lines.

4. Construct a mean proportional between two given straight lines.

5. Divide a given straight line into two such parts that the greater part shall be a mean proportional between the whole line and the other part; that is, into extreme and mean ratio. (Solution, page 300.)

6. Inscribe a regular decagon in a circle.

Divide the radius into extreme and mean ratio. The greater segment is the side of the decagon. (Solution, page 300.)

7. Inscribe a regular pentadecagon in a circle.

$$\frac{1}{6} - \frac{1}{10} = ?$$

8. Through a given point, in a given angle, draw a straight line so that the segments between the point and the sides of the angle shall be equal.

9. From the obtuse angle of a triangle, draw a line to the base which shall be a mean proportional between the segments into which it divides the base.

10. Three points being given, find a fourth point through which, if any straight line be drawn, perpendiculars upon it from two of the former shall together be equal to the perpendicular from the third.

11. Through a given point, between two non-parallel straight lines, to draw a third straight line, terminating in the former, which shall be bisected at the given point.

12. Through a given point draw a straight line, so that the parts of it intercepted between that point and perpendiculars drawn from two other given points may have a given ratio.

13. From a given point between two indefinite straight lines given in position, draw a line which shall be terminated by the given lines, and bisected in the given point.

14. If from a given point any number of straight lines be drawn to a straight line given in position, determine the locus of the points of section which divide them in a given ratio.

15. In any right-angled triangle, the straight line joining the right angle and the bisection of the hypotenuse is equal to half the hypotenuse.

16. If from any point within an equilateral triangle perpendiculars be drawn to the sides, they are together equal to a perpendicular drawn from any of the angles to the opposite side.

17. If the points of bisection of the sides of a given triangle be joined, the triangle so formed will be one-fourth of the given triangle.

18. If from the angles at the base of a triangle perpendiculars be let fall on a line which bisects the vertical angle, the part of this line intercepted between these perpendiculars will be bisected by a perpendicular from the middle of the base.

19. If from any point in the base of an isosceles triangle perpendiculars be drawn to the sides, these together shall be equal to a perpendicular drawn from either extremity of the base to the opposite side.

20. Bisect a given triangle by a line drawn from a given point in one of its sides.

21. If in two triangles the vertical angle of the one be equal to that of the other, and one other angle of the former be equal to the exterior angle at the base of the latter, the sides about the third angle of the former shall be proportional to those about the interior angle at the base of the latter.

22. If the sides of a triangle be cut proportionally, and lines be drawn from the points of section to the opposite angles, the intersections of these lines will be in the same line ; viz., that drawn from the vertex to the middle of the base.

23. In any triangle, the intersection of the perpendiculars drawn from the angles to the opposite sides, the intersection of the lines from the angles to the middle of the opposite sides, and the intersection of the perpendiculars from the middle of the sides, are all in the same straight line ; and the distances of those points from one another are in a given ratio.

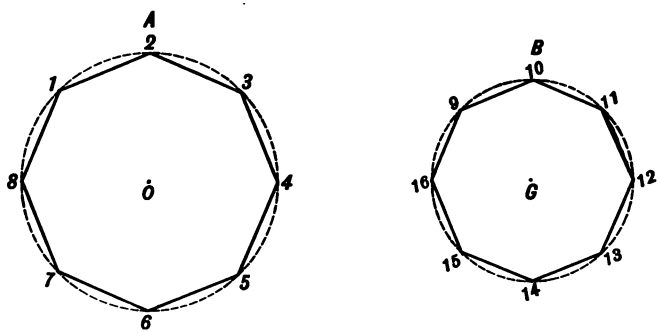
24. If from any angle of a rectangle a line be drawn to the opposite side, and from the adjacent angle of the trapezoid thus formed another be drawn perpendicular to the former, the rectangle contained by these two lines is equal to the given rectangle.

25. The rectangle of two sides of any triangle equals the rectangle of the perpendicular to the third side and the diameter of the circumscribed circle.

SECTION III.—POLYGONS.

PROPOSITION XI.

402. Theorem: ***Regular polygons of the same number of sides are similar.***



Statement: Regular polygons, as *A* and *B*, with the same number of sides are similar; that is, the sides are proportional, and the angles are equal.

Construction: Circumscribe circles. (BOOK IV., PROP. XXV.)

Demonstration: The sides are proportional, because, being equal, by definition of regular polygons, they form equal fractions or ratios.

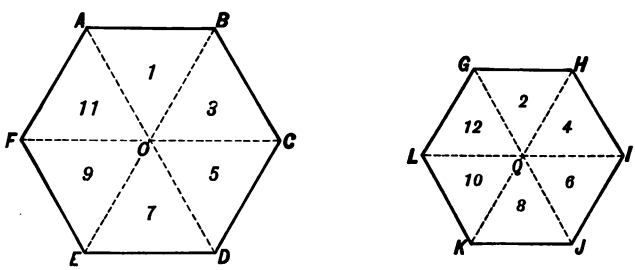
As the sides form equal chords, the arcs in each circle are equal. (BOOK IV., PROP. III.)

The angles are, then, all equal, as they have equal measures, the same fraction of a circumference. (BOOK VI., PROP. II.)

Conclusion: *A* and *B*, being any regular polygons, etc.

PROPOSITION XII.

403. Theorem: ***Similar polygons may be divided into the same number of similar triangles similarly placed.***



Statement: Similar polygons, *A B D* and *G H J*, may be divided into the same number of similar triangles similarly placed.

Construction: From *O*, any point in *A B D*, draw straight lines to the vertices. Construct *G H Q*, similar to *A B O*; and, from *Q* draw straight lines to the remaining vertices of *G H J*.

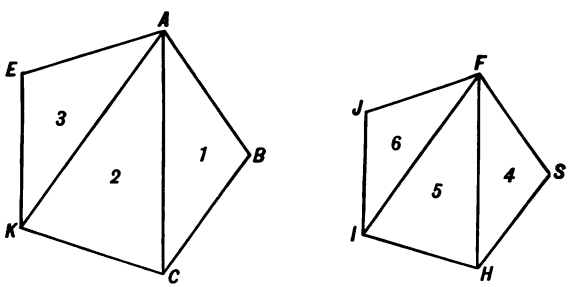
Demonstration: *A B : G H : : B O : H Q*, by construction; but, *A B : G H : : B C : H I*, by hypothesis; then, *B O : H Q : : B C : H I*. (BOOK V., PROP. XI.) The angles *O B C* and *Q H I* being equal, by equal subtractions, the triangles *O B C* and *Q H I* are similar. (PROP. III.)

In similar manner, the triangles on the corresponding sides may be proved respectively similar.

Conclusion: *A B D* and *G H J* being any similar, etc.

Proposition XIII.

404. Theorem: ***Polygons composed of the same number of triangles, similar each to each, and similarly placed, are similar.***



Statement: Let any polygons, *A B K* and *F S I*, contain the same number of triangles, *1, 2, 3* and *4, 5, 6,* similar, each to each, and similarly placed. The polygons are similar.

Demonstration: The angles of the corresponding triangles being equal, by hypothesis, the angles of the polygons are equal, each to each, being either corresponding angles of corresponding triangles, as *E* and *J*, or the sums of corresponding equal angles, as *A* and *F*.

A B : F S : : B C : S H, and *B C : S H : : A C : F H*, as *1* and *4* are similar; but, *C K : H I : : A C : F H*, *2* and *5* being similar; then,

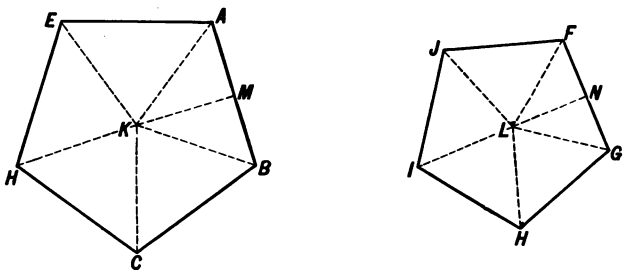
$$BC : SH :: CK : HI.$$ (Book v., Prop. xi.)

In similar manner, the other sides in succession may be proved to be proportional.

Conclusion: *A B K* and *F S I* being any polygons, etc.

PROPOSITION XIV.

405. Theorem: ***The perimeters of similar polygons are proportional to any homologous lines.***



Statement: Let *ABD* and *FGI* be any similar polygons, the lines drawn from *K* and *L* making similar triangles. The perimeters *ABCDE* and *FGHIJ* are proportional to any of the homologous lines.

Demonstration: By hypothesis,

$AB : FG :: BC : GH :: CD : HI :: DE : IJ :: EA : JF$; or,

$AB + BC + CD + DE + EA : FG + GH + HI + IJ + JF :: AB : FG$

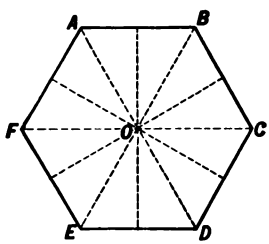
(BOOK V., PROP. XVI.), or, as any antecedent is to its consequent; that is, the perimeters are proportional to the corresponding sides of the polygons.

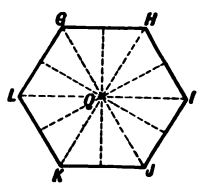
The sides of the polygon, however, are sides of the similar triangles, so that the perimeters, being proportional to the corresponding sides of the polygons, are proportional to the corresponding sides of the triangles, or any homologous lines. (BOOK V., PROP. XI.)

Conclusion: *ABCDE* and *FGHIJ* being any, etc.

PROPOSITION XV.

406. Theorem: ***Perimeters of regular polygons are proportional to the apothems, or to the radii, or the diameters of the inscribed or circumscribed circles.***





Statement: Let *A B D* and *G H J* be regular polygons. Their perimeters are proportional to the apothems, or to the radii, or the diameters of the inscribed or circumscribed circles.

Construction: Find the centers, and draw the inscribed and circumscribed circles; also, the apothems, and the straight lines from the centers to the vertices of the polygons.

Demonstration: The triangles thus constructed are all similar, as the corresponding angles are all equal. (BOOK IV, PROP. XXV.) As the apothems, and the radii and the diameters of the inscribed and circumscribed circles, are homologous lines, the perimeters are proportional to them. (PROP. XIV.)

Conclusion: *A B D* and *G H J* being any regular polygons, etc.

MISCELLANEOUS EXERCISES.

407. 1. The rectangle of any two sides of a triangle is equal to the rectangle of the altitude to the third side and the diameter of the circumscribed circle.

2. If from one of the angles of a parallelogram a straight line be drawn, cutting the diameter, a side, and a side produced, the segment intercepted between the angle and the diameter is a mean proportional between the segments intercepted between the diameter and the sides.

3. If two lines be drawn parallel and equal to adjacent sides of a parallelogram, the lines joining their extremities, if produced, will meet the diameter in the same point.

4. If two opposite angles of a trapezium be right angles, the angles subtended by either side at the two opposite angular points will be equal.

5. Inscribe a square in a given semicircle.

6. Determine the figure formed by joining the points of bisection of the sides of a trapezium, and its ratio to the trapezium.

7. Construct the square, the difference between the diameter and the side of the square being given.

8. Construct a regular pentagon, the length of one of its sides being given.

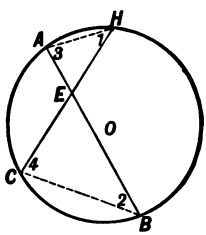
9. The sides of a triangle are 15, 18, and 20. If, in a similar triangle, the side corresponding to 15 is 5, what are the other sides of the similar triangle?

10. If a chord 10 inches long is 5 inches distant from the center of a circle, what is the radius and the distance from the end of the chord to the end of the radius perpendicular to the chord?

SECTION IV.—CIRCLES.

PROPOSITION XVI.

408. Theorem: ***The segments of intersecting chords in a circle are reciprocally proportional.***



Statement: Let the circle whose center is *O* be any circle, and *A B* and *C H* any two of its intersecting chords.

Prove that $AE : HE :: CE : BE$.

Construction: Draw the chords *B C* and *A H*.

Demonstration: The angles at *E* are equal (BOOK I., PROP. VII.); and *3* and *4* are equal, each being measured by half of the arc *B H*. (BOOK VI., PROP. II.) The triangles *A H E* and *B C E* are, therefore, similar (PROP. I.), and, therefore, the sides opposite the equal angles are proportional; thus,

$AE : EC :: EH : EB$; that is,

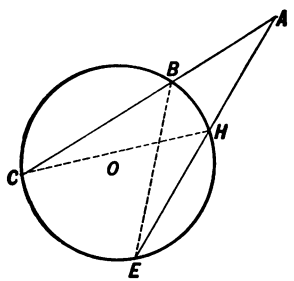
the segments are reciprocally proportional.

Conclusion: The circle whose center is *O* being, etc.

Corollary I. The rectangle of the segments of chords passing through a common point is a constant quantity.

PROPOSITION XVII.

409. Theorem: ***Secants drawn from the same point without a circle are reciprocally proportional to their external segments.***



Statement: Let *A B C* and *A H E* be any secants drawn from the same point *A*, without any circle.

Prove *A C : A E : : A H : A B.*

Construction: Draw the chords *C H* and *B E.*

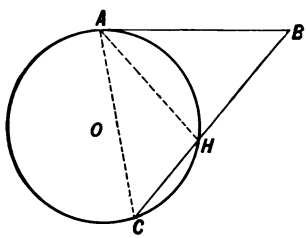
Demonstration: The two triangles *A C H* and *A B E* are similar; for the angle *A* is common, and the angles *C* and *E* are equal, each being measured by half of *B H.* (PROP. I., COR. I.) The corresponding sides are, therefore, proportional; that is, the sides opposite the equal angles are proportional; thus, *A C : A E : : A H : A B.*

Conclusion: The circle whose center is *O* being, etc.

Corollary I. The rectangle of the external segments and the secants drawn from the same point is constant.

PROPOSITION XVIII.

410. Theorem: ***If from any point without a circle, a tangent and secant are drawn, the tangent is a mean proportional between the secant and its external segment.***



Statement: Let *A B* and *B C* be respectively any tangent and secant drawn from the same point, *B*, without any circle. Prove, *B C : A B : : A B : B H.*

Construction: Draw the chords *A C* and *A H.*

Demonstration: In the triangles *A B C* and *A B H*, the angle *B* is common; and *B A H* is equal to *A C H*, each being measured by half of the arc *A H.* (PROPS. II. and III.) The triangles *A B C* and *A B H* are, then, similar (PROP. I., COR. I.); and

B C : A B : : A B : B H,

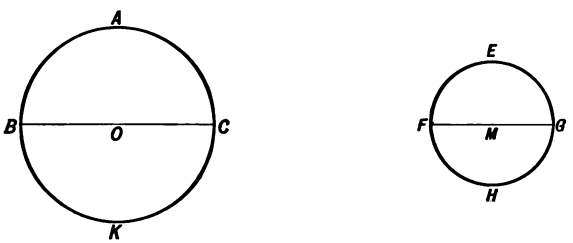
the tangent being a mean proportional between the secant and its external segment.

Conclusion: The circle whose center is *O* being, etc.

Corollary I. The rectangle of any secant and its external segment equals the square of the tangent from the same point.

PROPOSITION XIX.

411. Theorem: ***Circumferences of circles are proportional to the diameters or to the radii.***



Statement: Let $ABKC$ and $EFHG$ be any two circles. The circumference $ABKC$ is to the circumference $EFHG$ as the diameter BC is to the diameter FG; or, as the radius OC is to the radius MG.

Demonstration: Expressing the relation between the circumferences and the diameters,

$$ABKC = \pi BC; \text{ and, } EFHG = \pi FG \text{ (BOOK VI., PROP. XXI.)};$$

then, $\pi = \frac{ABKC}{BC}$; and, $\pi = \frac{EFHG}{FG}$. Therefore,

$$ABKC : BC :: EFHG : FG; \text{ or, } ABKC : EFHG :: BC : FG$$

(BOOK V., PROP. II.); that is, the circumferences are proportional to the diameters. *Q. E. D.*

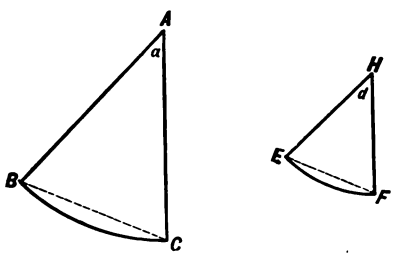
Since, $BC : FG :: OC : MG$ (BOOK V., PROP. XV., COR.),

$$ABKC : EFHG :: OC : MG;$$ that is, the circumferences are proportional to the radii. *Q. E. D.*

Conclusion: $ABKC$ and $EFHG$ being any circles, etc.

PROPOSITION XX.

412. Theorem: ***Arcs of similar sectors of circles are proportional to the radii or the diameters of the circles; or, to the chords of the arcs.***



Statement: Let BAC and EHF be similar sectors. The arcs BC and EF are proportional to the radii AB and HE, or to the diameters; or, to the chords BC and EF.

Demonstration:

$BC : 2\pi AB :: a : 360$, and $EF : 2\pi HE :: d : 360$;

then, $BC = \dfrac{2\pi \times AB \times a}{360}$; and $EF = \dfrac{2\pi \times HE \times d}{360}$

(BOOK VI., PROP. I.); hence, $\dfrac{BC}{AB} = \dfrac{2\pi a}{360}$; and, $\dfrac{EF}{HE} = \dfrac{2\pi d}{360}$.

The angles a and d being equal, by definition,

$$\frac{BC}{AB} = \frac{EF}{HE};$$ and, $BC : AB :: EF : HE$.

The arcs are, then, also proportional to the diameters. (BOOK V., PROP. XV., COR.) The triangles ABC and HEF are, however, similar (PROP. I., COR. II.); and, therefore, the radii are proportional to the chords. The arcs are, then, also proportional to the chords. (BOOK V., PROP. XI.)

Conclusion: BAC and EHF being any similar sectors, etc.

Scholium I. In similar manner, it may be proved that the arcs of similar segments are proportional to the chords, or to the radii or diameters of the circles.

MISCELLANEOUS EXERCISES.

413. 1. If a line, tangent to two circles, cuts another line joining their centers, the segments of the latter will be to each other as the diameters of the circles.

2. If, from a point in the circumference of a circle, any number of chords be drawn, the locus of their points of bisection will be a circle.

3. If, from the extremities of any chord in a circle, perpendiculars be drawn, meeting a diameter, the points of intersection are equally distant from the center.

4. If, from the extremities of the diameter of a semicircle, perpendiculars be let fall on any line cutting the semicircle, the parts intercepted between those perpendiculars and the circumference are equal.

5. In a given circle, to construct a chord parallel to a given chord, and having a given ratio to it.

6. From a given point in the diameter of a semicircle produced, to draw a line cutting the semicircle so that lines drawn from the points of intersection to the extremities of the diameter, cutting each other, may have a given ratio.

7. If two circles are tangent to each other, externally or internally, any straight line drawn through the point of contact will cut off similar segments.

8. If two circles are tangent to each other, externally or internally, any two straight lines drawn through the point of contact, and terminated both ways by the circumference, will be cut proportionally by the circumference.

9. If two circles are tangent to each other externally, and parallel diameters be drawn, the straight line joining the extremities of these diameters will pass through the point of contact.

10. If two circles are tangent to each other and to a straight line, the part of the line between the points of contact is a mean proportional between the diameters of the circles.

11. If two circles are tangent to each other internally, and any two perpendiculars to their common diameter be produced to cut the circumferences, the lines joining the points of intersection and the point of contact are proportional.

12. If, from any point in the diameter of a circle produced, a tangent be drawn, a perpendicular from the point of contact to the diameter will divide it into segments which have the same ratio that the distances of the point without the circle from each extremity of the diameter have to each other.

13. If, from the extremities of the diameter of a circle, tangents be drawn, any other tangent to the circle, terminated by them, is so divided at the point of contact that the radius of the circle is a mean proportional between its segments.

14. Through a given point within a given circle, to draw a straight line such that the parts of it intercepted between that point and the circumference may have a given ratio.

15. A straight line being divided in two given points, find a third straight line, such that its distances from the extremities may be proportional to its distances from the given points.

16. In a straight line given in position, find a point at which two straight lines drawn from given points on the same side will contain the greatest angle.

17. Divide a straight line into parts, such that their rectangle may be equal to the square of their difference.

18. Draw two lines such that the sum of their squares may equal a given square, and their rectangle, a given rectangle.

19. Divide a straight line into two parts, so that the rectangle contained by the whole and one of the parts may equal the square of a line given less than the line to be divided.

20. Divide a given line into two such parts that the rectangle contained by the whole line and one of the parts may be m times the square of the other part, m being any number.

21. Divide a given line into such parts that the square of the one shall be equal to the rectangle contained by the other and a given line.

22. A straight line being given in magnitude and position, draw to it, from a given point, two lines whose rectangle shall be equal to a given rectangle, and which shall cut off equal segments from the given line.

23. From any angle of a triangle not isosceles, about the given angle, draw a line without the triangle to the opposite side produced, which shall be a mean proportional between the segments of the side.

24. Bisect a triangle by a line parallel to one of its sides.

25. Divide a given triangle into any number of parts, having a given ratio to each other, by lines drawn parallel to one of the sides of the triangle.

26. Construct a rectangle which shall equal a given square, and have its adjacent sides together equal to a given line.

27. Construct a rectangle which shall equal a given square, and have the difference of its sides equal to a given line.

28. If the radius of a circle be cut in extreme and mean ratio, the greater segment will equal the side of an equilateral and equiangular decagon inscribed in the circle.

29. If, through the middle point of a chord of a circle, two chords be drawn, lines joining their extremities will intersect the first chord at equal distances from the middle point.

30. If, from the vertex of an isosceles triangle, a circle be described, with a radius less than one of the equal sides but greater than the perpendicular, the parts of the base cut off by it will be equal.

31. If, in a right-angled triangle, a perpendicular be drawn from the right angle to the hypotenuse, and circles inscribed in the triangles on each side of it, their diameters will be proportional to the subtending sides of the right-angled triangle.

32. If three circles are tangent to each other, two of which are equal, the vertical angle of the triangle formed by joining the points of contact equals either of the angles at the base of the triangle, formed by joining their centers.

33. If three equal circles are tangent to one another, compare the area of the triangle formed by joining their centers with that formed by joining the points of contact.

34. If the diameter of a semicircle be divided into any number of parts, and on them semicircles be described, their circumferences will together be equivalent to the circumference of the given semicircle.

35. Construct a triangle, the vertical angle, the ratio of its sides, and the diameter of the circumscribing circle given.

36. If the diagonals of a quadrilateral divide one another so that the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other, then a circle may be circumscribed about the quadrilateral.

37. Construct a right-angled triangle whose perimeter is given and whose sides are in geometrical progression.

38. If two opposite sides of a quadrilateral be prolonged to meet, and if it be found that the rectangle contained by one of the lines thus produced, and the part produced, equal the rectangle contained by the other line and the part produced, then the vertices of the quadrilateral are concyclic.

39. The sides of a pedal triangle are equally inclined to the sides of the original triangle which they meet.

40. The three other parts of the original triangle cut off by the sides of the pedal triangle are triangles mutually equiangular with one another and the original triangle.

41. The three circles, the circumferences all passing through the orthocenter, and each through the extremities of one side of a triangle, are equal to one another and the given triangle.

42. The radius of a circle is 7 inches. If a tangent is 15 inches long, what is the length of the secant from the same point, the secant being 5 inches from the center?

43. Find the radius of a circle to which a tangent 12 inches long is drawn, and from the same point, a secant whose internal segment, or chord, is 5 inches long.

44. If three or more straight lines cut two parallels proportionally, they pass through a common point.

45. Find the side of a square, which is equal to a rectangle whose sides are 5 and 7.

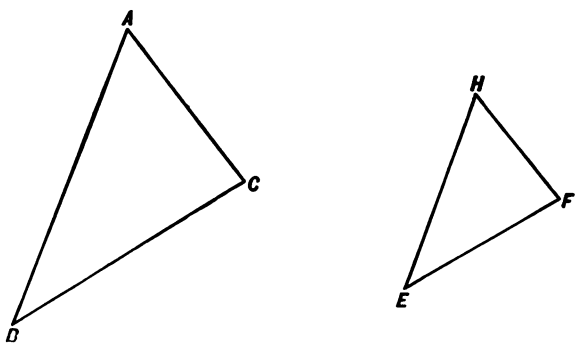
46. Find the length of the longest and shortest chords that can be drawn through a point 7 inches from the center of a circle with a radius of 15 inches.

47. Find the length of a chord joining the points of contact of two tangents drawn from a point 12 inches from the center of a circle with a radius of 8 inches.

SECTION V.—PROPORTIONAL AREAS.

PROPOSITION XXI.

414. Theorem: ***Similar triangles are proportional to the squares of their homologous sides.***



Statement: Let ABC and HEF be any two similar triangles. Their areas are proportional to the squares of their corresponding sides; thus, $ABC : HEF :: AB^2 : DE^2$.

Demonstration: The triangles being similar, the sides are proportional; thus, $AB : HE :: AC : HF$. Multiplying by the proportion, $AB : HE :: AB : HE$ (BOOK V., PROP. XIV.),

$$AB^2 : HE :: AB \times AC : HE \times HF.$$

Since the triangles are similar, the corresponding angles, as A and H, are equal; and, therefore (BOOK VI., PROP. VII.),

$$ABC : HEF :: AB \times AC : HE \times HF.$$

Comparing the last two proportions, it is evident that

$$ABC : HEF :: AB^2 : HE^2$$

(BOOK V., PROP. XI.); that is, the triangles are proportional to the squares of two corresponding sides.

The lines AB and HE are, however, proportional to any other corresponding sides, or homologous lines; and, therefore, $AB^2 : HE^2$ equals the ratios of the squares of any other homologous sides or lines. (BOOK V., PROP. IX.) By substitution, then, $ABC : HEF$ as the squares of any other homologous sides or lines. (BOOK V., PROP. XI.)

Conclusion: ABC and HEF being any two similar, etc.

Exercises. 1. What is the area of a triangle similar to a triangle whose area is 25 square inches, one side of the former being to a side of the latter as 2 : 1?

2. Prove that $R^2 = P^2 - D^2$, where R is the radius, and P and D are the sides, respectively, of a regular inscribed pentagon and decagon.

3. Construct a triangle which shall contain four times the area of a triangle whose sides are 5, 6, and 7. What is the length of the sides?

4. Find the sides of a triangle similar to the triangle whose sides are 8, 9, and 10, and whose area shall be nine times the area of the latter triangle.

5. In any equilateral triangle, inscribe three equal circles, tangent to each other and the given circle.

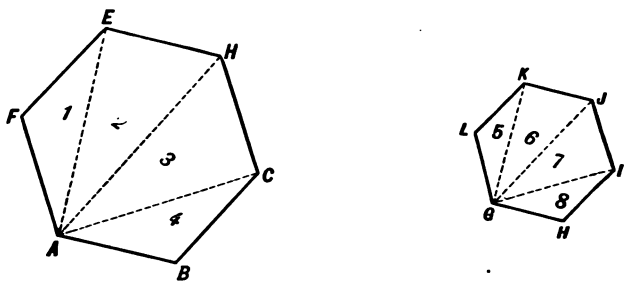
6. The sum of the squares of the diagonals of any quadrilateral is twice the sum of the squares of the diameters.

7. If two homologous sides of similar triangles are as 4 : 5, what is the ratio of their areas?

8. What is the ratio of the areas of two similar triangles whose sides are as 2 : 3?

PROPOSITION XXII.

415. Theorem: ***Similar polygons are proportional to the squares of any homologous lines.***



Statement: Similar polygons, as *A B D* and *G H J*, are proportional to the squares of any homologous lines, as the squares of *A F* and *G L*.

Construction: Draw the diagonals *A C*, *A D*, and *A E*, and also *G I*, *G J*, and *G K*, from corresponding angles, *A* and *G*. Number the triangles as in the diagram.

Demonstration: The polygons have been divided into the same number of similar triangles similarly placed. (PROP. XII.) Then, $1 : 5 :: AE^2 : GK^2$ (PROP. XXI.);

and, also, $2 : 6 :: AE^2 : GK^2$ (PROP. XXI.);

and, therefore, $1 : 5 :: 2 : 6$. (BOOK V., PROP. XI.)

In similar manner, it may be shown that

$$2 : 6 :: 3 : 7, \text{ and } 3 : 7 :: 4 : 8;$$

and, thus, $1 : 5 :: 2 : 6 :: 3 : 7 :: 4 : 8$ (BOOK V., PROP. XI.);

and, also, $1 + 2 + 3 + 4 : 5 + 6 + 7 + 8 :: 1 : 5$. (BOOK V., PROP. XVI.)

The triangles *1, 2, 3,* and *4,* however, together equal the polygon *A B C D E F;* and the triangles *5, 6, 7,* and *8,* the polygon *G H I J K L.* (AXIOM. II.) By substitution, therefore,

$$ABCDEF : GHIJKL :: 1 : 5 ;$$

but, $1 : 5 :: AF : GL^2$ (PROP. XXI.);

therefore, $ABCDEF : GHIJKL :: AF^2 : GL^2$ (BOOK V., PROP. XI.); that is, the polygons are proportional to the squares of the corresponding sides. As the corresponding sides are proportional to any homologous lines, the squares of any two homologous lines may be substituted for $AF^2 : GL^2$. (BOOK V., PROP. XI.)

Conclusion: *A B D* and *G H J* being any two similar, etc.

Corollary I. In similar polygons, homologous triangles are like parts of the polygons to which they belong.

Exercises. 1. The square of the bisectrix of any angle of a triangle is equal to the rectangle of the sides of the given angle diminished by the rectangle of the segments of the third side.

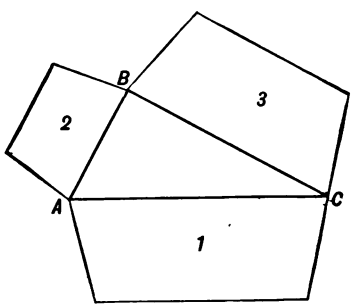
2. The bisectrices of the exterior angles of a triangle form a triangle whose angles are each equal to the half of two of the angles of the given triangle.

3. The line joining the middle point of the diagonals of a quadrilateral is bisected by the lines joining the diameters of the quadrilateral.

4. Prove that five times the square of the hypotenuse of a right-angled triangle is equal to four times the sum of the squares of the medians from its extremities.

PROPOSITION XXIII.

416. Theorem: ***Of three similar figures constructed on the hypotenuse and the sides of a right-angled triangle, the figure constructed on the hypotenuse is equal to the sum of the two other figures.***



Statement: Let ABC be any right-angled triangle; and *1*, *2*, and *3*, similar figures. Prove *1* equal to *2* and *3*.

Demonstration: $1 : AC^2 :: 2 : AB^2 :: 3 : BC^2$ (PROP. XXII.); then $1 : AC^2 :: 1 + 2 + 3 : AC^2 + AB^2 + BC^2$ (BOOK V., PROP. XVI.); and, $1 : AC^2 :: 2 + 3 : AB^2 + BC^2$. (BOOK V., PROP. V.)

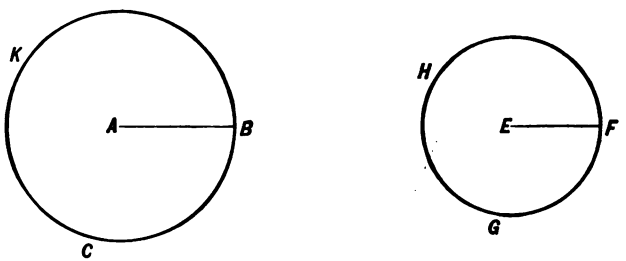
As *1* is equal to the sum of *2* and *3* (BOOK III., PROP. XXIV.), AC^2 is equal to the sum of AB^2 and BC^2.

Conclusion: ABC being any right-angled triangle, etc.

Corollary I. The figure constructed on either side of a right-angled triangle is equal to the difference between similar figures constructed on the hypotenuse and the other side.

PROPOSITION XXIV.

417. Theorem: ***The areas of circles are proportional to the squares of their radii.***



Statement: Let the circles BCK and FGH be any two circles. Prove the areas proportional to AB^2 and EF^2.

Demonstration: Area $BKC = \pi \times AB^2$, and $FGH = \pi \times EF^2$ (BOOK VI., PROP. XXII.); therefore,

$$\pi = \frac{\text{area } BKC}{AB^2}, \text{ and, } \pi = \frac{\text{area } FGH}{EF^2};$$

then,
$$\frac{\text{area } BKC}{AB^2} = \frac{\text{area } FGH}{EF^2}; \text{ or,}$$

$$\text{area } BKC : \text{ area } FGH :: AB^2 : EF^2.$$

Conclusion: BKC and FGH being any two circles, etc.

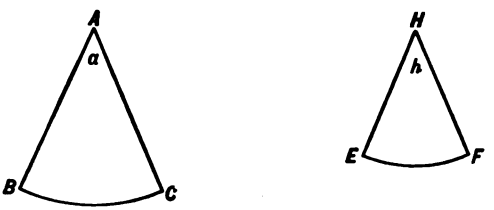
Corollary I. The areas of circles are proportional to the squares of any homologous lines in the circles.

Exercises. 1. Find area, when radius is 5 inches.

2. Find area, when diameter equals 20 inches.

PROPOSITION XXV.

418. Theorem: ***The areas of similar sectors of circles are proportional to the squares of the radii or chords, or any homologous lines of the circles.***



Statement: Let ABC and HEF be similar sectors of circles; that is, they have equal angles, a and h. Prove ABC and HEF proportional to the squares of AB and HE, or of the chords, BC and EF; or, of any homologous lines.

Demonstration: Area $ABC = \frac{\pi \times AB^2 \times a}{360}$, and (BOOK VI., PROP. XXIII.), area $HEF = \frac{\pi \times HE^2 \times h}{360}$; then,

$$\pi = \frac{\text{area } ABC \times 360}{a \times AB^2}, \text{ or, } \frac{\text{area } HEF \times 360}{h \times AH^2};$$

as $a = h$, area ABC : area HEF : : AB^2 : HE^2. *Q. E. D.*

The radii are, however, proportional to the chords, the diameters, or any homologous lines (PROP. XX.); and, the squares of these lines are proportional. (BOOK V., PROP. XIX.) The sectors are, then, proportional to the squares of any homologous lines. (BOOK V., PROP. XI.)

Conclusion: ABC and HEF being any similar sectors, etc.

Scholium. In similar manner, it may be proved that similar segments are proportional to the squares of their chords or radii or diameters, or any homologous lines of the circles.

MISCELLANEOUS EXERCISES.

419. 1. A trapezium is divided by its diagonals into four triangles proportional to one another.

2. If, from the three angles of a triangle, lines be drawn to the points of bisection of the opposite sides, the squares of the distances between the angles and the common intersection are together one-third of the squares of the sides of the triangle.

3. If, from any point within or without any rectilineal figure, perpendiculars be let fall on every side, the sum of the squares of the alternate segments made by them will be equal.

4. The squares of the diagonals of a trapezium are together double the squares of the two lines joining the bisections of the opposite sides.

5. If, on any two segments of the diameter of a semicircle, semicircles be described, the area included between the three circumferences will be equal to the area of a circle whose diameter is a mean proportional between the segments.

6. If the diagonals of an equilateral and equiangular pentagon be drawn to cut one another, the greater segments will be equal to the side of the pentagon; and the diagonals cut each other in extreme and mean ratio.

7. Construct a square equal to the sum of two squares.

8. Construct a square equal to the difference of two squares.

9. In a triangle whose sides are 7, 8, and 9 inches, find the area ; the altitude ; the median to 7 ; the bisectrix to 8.

10. A cane, 3 feet long, when in a vertical position, casts a shadow 1 foot 6 inches long. How high is a steeple which at the same time casts a shadow 75 feet long ?

11. If the sides of a triangle are 7, 5, and 8, and the side corresponding to 5, in a similar triangle, is 25, how long are its other sides ?

12. Of an equilateral triangle, find the side, when the altitude is 4 ; find the altitude, when the side is 4.

13. The sides of a triangle are 4.5, and 6 ; find the segments of 4 made by a line, 4.5, parallel to 6.

14. If the sides of a triangle are 3, 5, and 6, what is the length of a line parallel to 6 that divides 3 into two parts, as 2 : 1, the shorter segment being nearer 6 ?

15. What is the area of the rectangle that can be made by the segments of any chord, 3 inches from the center of a circle whose diameter is 20 inches ?

16. When a ladder, 30 feet long, touches the side of a house at a point 25 feet from the ground, how far from the house is the lower end ?

17. The radius of a circle is 5 inches ; what is the length of the tangents drawn from a point 10 inches from the center ? 15 inches ?

18. How far from the center are the chords joining the points of tangency in each part of the last exercise ?

19. If one side of a polygon is 2 feet 6 inches long, what is the length of the corresponding side of a similar polygon, the perimeters being, respectively, 20 feet and 30 feet ?

20. Find the side of a regular decagon inscribed in a circle whose radius is 12 inches.

NUMERICAL EXERCISES.

420. 1. Find medians of a triangle, in terms of sides.

$m = \frac{1}{2}\sqrt{2(a^2 + b^2) - c^2}$; m being the median to c.

2. Find the line that bisects an angle of a triangle, in terms of the sides.

$x = \frac{2}{a + b}\sqrt{a\,b\,c\,(s - c)}$, x being the bisectrix to c.

3. Find radius of the circumscribed circle, in terms of sides.

$$R = \frac{a\,b\,c}{4\sqrt{s\,(s - a)\,(s - b)\,(s - c)}}.$$

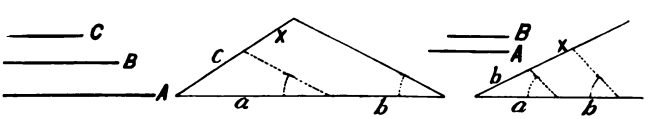
4. Find area, in terms of radius of circumscribing circle.

$$\text{Area} = \frac{a\,b\,c}{4\,R}.$$

CONSTRUCTIONS.

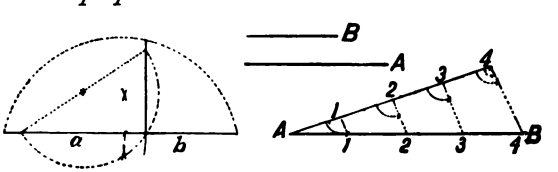
421. Prove the following constructions:

1. *A fourth proportional to three lines.*



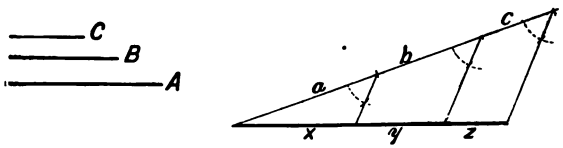
2. *A third proportional to two lines.*

3. *A mean proportional to two lines.*

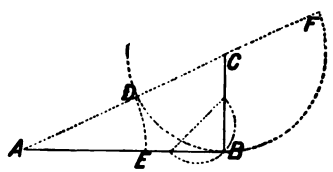


4. *A straight line, equal parts.*

5. A straight line, proportional parts.



6. A straight line, extreme and mean ratio ; that is, so that entire line is to greater segment, as greater segment is to less segment.



BC is equal to $\frac{1}{2}$ **AB**. **AD** = **AE**.

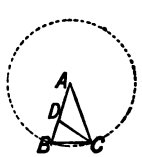
AF : AB : : AB : (AD, or) **AE** (BOOK VII., PROP. XVIII.) ; and, then (BOOK V., PROP. V.), (**AF** − **AB,** or) **AE : AB** : : (**AB** − **AE,** or) **EB : AE ;** inverting, **AB : AE : : AE : EB.**

Q. E. D.

7. A regular decagon, inscribed in a given circle.

Divide radius into extreme and mean ratio ; thus,

AB : AD : : AD : DB.



Const.—Make **BC** equal to **AD ;** and, draw **DC.**

Dem.—By substitution, **AB : BC : : BC : DB.** The △ **DBC** is, then, similar to △ **ABC ;** that is, it is isosceles ; and, **DC** equals **BC** and **AD.**

Ext. ∠ **BDC** = 2 × ∠ **DAC ;** ∴ ∠ **DBC** = 2 × ∠ **DAC ;** ∴ ∠ **ACB** = 2 × ∠ **DAC.** ∴ ∠ **DAC** = $\frac{1}{5}$ of two right angles or $\frac{1}{10}$ of four right angles, and **BC** is the chord of an inscribed regular decagon.

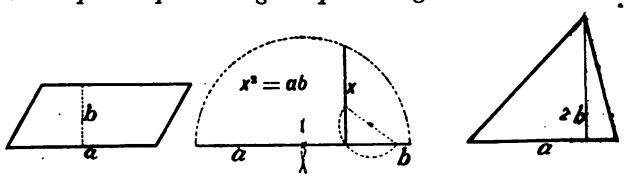
Q. E. F.

8. A regular pentagon inscribed in a given circle.

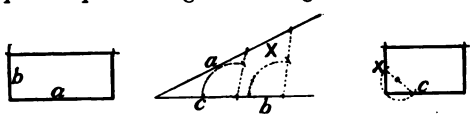
Construct decagon ; join alternate vertices.

9. A regular inscribed pentadecagon. $\left(\frac{1}{6} - \frac{1}{10} = \frac{1}{15}\right)$

10. A square equal to a given parallelogram.



11. A square equal to a given triangle.



12. On a given line, a rectangle equal to a given rectangle.

BOOK VIII.—PLANES.

SYNOPSIS.

SECTION I.—ONE PLANE.

422. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. I. If two straight lines are perpendicular to the same plane, they are parallel.

PROP. II. From a given point, only one perpendicular to a plane can be drawn.

PROP. III. Of lines drawn from the same point to the same plane (I.), the perpendicular is the shortest; (II.) oblique lines meeting the plane at points equally distant from the foot of the perpendicular are equal; and (III.), of oblique lines meeting the plane at points unequally distant from the foot of the perpendicular, that which meets the plane at the greater distance is the longer line.

PROP. IV. If a straight line is perpendicular to two straight lines at their point of intersection, it is perpendicular to the plane of those lines.

PROP. V. If one of two parallel straight lines is perpendicular to a plane, the other, also, is perpendicular.

3. MISCELLANEOUS EXERCISES.

SECTION II.—TWO PLANES.

423. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. VI. The intersection of two planes is a straight line.

PROP. VII. If a straight line is perpendicular to a plane, every plane passed through that line is perpendicular to the plane.

PROP. VIII. If a straight line is perpendicular to one of two parallel planes, it is perpendicular also to the other plane.

PROP. IX. Planes perpendicular to the same straight line are parallel.

PROP. X. Parallel lines between parallel planes are equal.

3. MISCELLANEOUS EXERCISES.

SECTION III.—SEVERAL PLANES.

424. 1. PROPOSITIONS.

PROP. XI. Lines intersecting parallel planes are divided into proportional segments.

PROP. XII. If two intersecting planes are perpendicular to a third plane, their intersection is perpendicular to that plane.

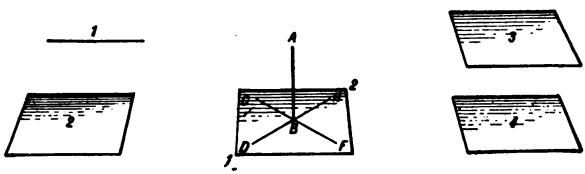
PROP. XIII. The sum of any two of the three plane angles formed by the edges of a trihedral angle are together greater than the third angle.

PROP. XIV. If the plane angles of two trihedral angles are equal, each to each, the planes of the equal angles are equally inclined to each other.

PROP. XV. The sum of the plane angles made by the edges of any convex polyhedral angle is less than four right angles.

2. MISCELLANEOUS EXERCISES.

SECTION I.—ONE PLANE.



DEFINITIONS.

425. A **plane** is a surface such that, if any two of its points are joined by a straight line, that line will lie wholly in the surface.

426. A line is **perpendicular** to a plane when it is perpendicular to every line in the plane drawn through its foot.

The line ***A B*** is perpendicular to the plane ***12;*** it is perpendicular to ***C F*** and ***S H,*** lines in the plane ***12*** drawn through ***B,*** the foot of the perpendicular. The plane is perpendicular to the line.

427. A line or a plane is **parallel** to a plane when it cannot touch that plane, how far soever in any direction either may be produced. The plane is, then, parallel to the line.

428. A plane is determined by the following:

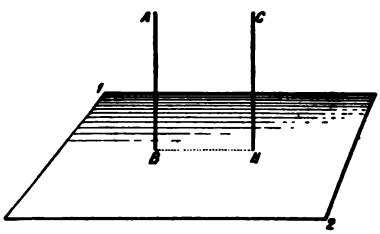
1. Three points not in the same straight line;
2. A straight line and a point without the line;
3. Two straight lines that intersect;
4. Two parallel straight lines.

Pass a plane through any two points, and turn it on the line joining these points, as an axis, until it includes the third point. Any motion, in either direction, will cause the plane to contain no longer the third point. The four statements are different expressions of the same facts.

429. A **projection** of a **point** on a plane is the foot of the perpendicular from the point to the plane; the **projection** of a **line** on a plane is the locus of the projections of its points.

PROPOSITION I.

430. Theorem: ***If two straight lines are perpendicular to the same plane, they are parallel.***



Statement: Let *A B* and *C H* be perpendicular to *1 2*. Prove that they are parallel lines.

Construction: Draw *B H*.

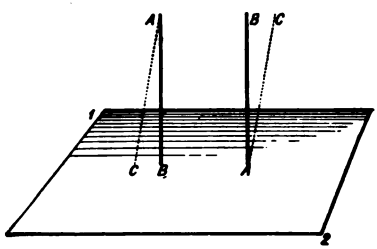
Demonstration: *A B H* and *C H B* are right angles, by hypothesis, and by the definition of perpendiculars to a plane. *A B* and *C H* being perpendicular to *B H*, a straight line in the plane, they are parallel to each other. (BOOK I., PROP. XIII.)

Conclusion: *1 2* being any plane, and *A B* and *C H* being, etc.

Corollary I. All straight lines perpendicular to a plane are parallel to one another.

PROPOSITION II.

431. Theorem: ***From a given point only one perpendicular can be drawn to a plane.***



Statement: From any given points, as *A*, either in a given plane or without it, only one perpendicular, *A B*, can be drawn to the plane *1 2*.

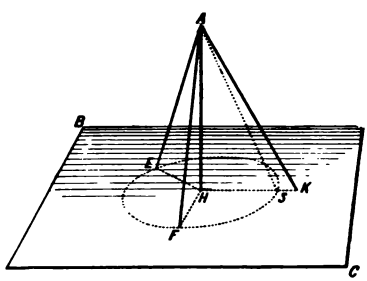
Construction: If possible, through either point *A*, draw a second perpendicular, *A C*.

Demonstration: If *A B* and *A C* be both perpendicular to *1 2*, they are both parallel to every other perpendicular to the plane, as *B A*. (PROP. I.) This cannot be true, as, then, there would be two straight lines drawn through the same point parallel to the same straight line. (POST. III.) A second perpendicular cannot, then, be drawn; that is, only one perpendicular to the plane *1 2* can be drawn through any given point, *A*.

Conclusion: *1 2* being any plane, and *A* being any point, etc.

Proposition III.

432. Theorem: ***Of lines drawn from the same point to the same plane, I. The perpendicular is the shortest; II. Oblique lines meeting the plane at points equally distant from the foot of the perpendicular, are equal; and, III. Of oblique lines meeting the plane at points unequally distant from the foot of the perpendicular, that which meets the plane at the greater distance is the longer line.***



Statement: Let *A* be a given point without the plane of *B C*, and let *A H* be the perpendicular to the plane. *A H* is the shortest line from the point *A* to the plane *B C*. If *H F* and *H E* are equal, the oblique lines *A F* and *A E* are equal. If *H K* is greater than *H F*, *A K* is longer than *A F*.

Construction: Take *H S*, a part of *H K*, equal to *H F*, and draw *A S*.

Demonstration: I. *A H*, the perpendicular, is the shortest line; for, if not, suppose that *A F* were shorter. This is impossible; for, *A H*, being perpendicular to the plane, is perpendicular to *F H* in the plane; *A H* is, therefore, less than *A F*.

(BOOK I., PROP. XXI.) In the same manner, it may be shown that no other line from *A* to the plane *B C* is shorter than *A H*, the perpendicular. The perpendicular is, then, the shortest distance from the given point to the given plane.

II. Oblique lines, as *A F* and *A E*, meeting the plane at points equally distant from the foot of the perpendicular, are equal. The triangles *A H F* and *A H E* are right-angled triangles. *A H* is common, and *E H* and *H F* are equal, by hypothesis. *A E* and *A F* are, therefore, equal (BOOK II., PROP. I.); but *A E* and *A F* are oblique lines meeting the plane at equal distances from the foot of the perpendicular.

III. The oblique line meeting the plane at the greater distance from the foot of the perpendicular is the longer line. *A F* and *A S* are equal (BOOK II., PROP. I.); but *A K* is longer than *A S*. (BOOK I., PROP. XXIII.) *A K* is, therefore, longer than *A F*.

Conclusion: *B C* being any plane, and *A* being any point, etc.

Corollary I. The shortest line from a given point to a given plane is the perpendicular from the point to the plane.

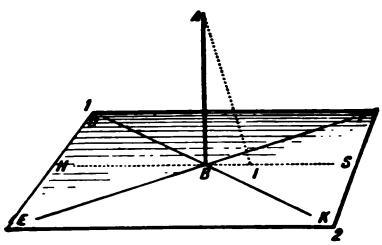
Corollary II. Equal oblique lines drawn from the same point to the same plane meet the plane at equal distances from the foot of the perpendicular.

Corollary III. The longer oblique line meets the plane at the greater distance from the foot of the perpendicular.

Exercise. Find the locus of the farther extremities of equal straight lines from a given point to a given plane.

PROPOSITION IV.

433. Theorem: ***If a straight line is perpendicular to two straight lines at their point of intersection, it is perpendicular to the plane of those lines.***



Statement: Let *A B* be perpendicular to *C K* and *E F* at *B*. Prove that *1 2* is perpendicular to the plane of the lines.

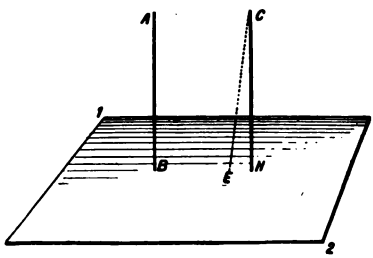
Construction: Draw *H S*, any line in the plane *1 2* passing through *B*, the foot of the perpendicular.

Demonstration: To be perpendicular to the plane, *A B* must be perpendicular to every line of the plane passing through *B*, its foot. *A B* is perpendicular to *C K* and *E F*, by hypothesis. *A B* is also perpendicular to *H S*; for, if it is not, draw *A I* perpendicular to *H S*. *A I* is, then, shorter than *A B* (BOOK I., PROP. XXI.); but, *A B*, being perpendicular, is the shortest line from *A* to the plane. (PROP. III.) *A I* cannot be shorter than *A B*. As *A B* is, then, perpendicular to *H S*, any line in the plane and passing through its foot, it is perpendicular to every line passing through its foot, and it is, therefore, perpendicular to the plane.

Conclusion: *A B* being any line perpendicular, etc.

PROPOSITION V.

434. Theorem: ***If one of two parallel straight lines is perpendicular to a plane, the other is also perpendicular to the plane.***



Statement: If *A B*, one of two parallel straight lines, *A B* and *C H*, is perpendicular to the plane *1 2*, the other, *C H*, is also perpendicular to *1 2*.

Construction: If *C H* is not perpendicular to *1 2*, draw *C E* perpendicular to *1 2*.

Demonstration: The lines *C E* and *A B* are parallel (PROP I.); and, *C H* and *C E* are both parallel to *A B*, which is impossible, as through a given point only one straight line can be drawn parallel to a given line. (POST III.) The supposition that *C H* is not perpendicular to *1 2* should be abandoned. *C H* is, then, perpendicular to *1 2*.

Conclusion: *A B* and *C H* being any two parallel lines, etc.

Corollary. If one of any number of parallel straight lines is perpendicular to a plane, the others are also perpendicular to that plane.

MISCELLANEOUS EXERCISES.

435. 1. If a plane bisects a straight line at right angles, every point in the plane is equally distant from the extremities of the line.

2. A straight line and a plane perpendicular to the same straight line are parallel.

3. Find the locus of the points in space, any one of which is equally distant from all points of the circumference of a circle.

4. If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

5. If, through one of the diagonals of a parallelogram, we pass any plane, the perpendiculars let fall from the extremities of the other diagonal on this plane will be equal.

6. Find the locus of all the points, in a given plane, which are at a given distance from a given point without the plane.

7. Find the locus of all the points, any one of which is equidistant from three given points.

8. Find the locus of the points in a given plane, the difference of the squares of the distances of which from two given points without the plane is constant.

9. In a given plane, and through a given point in this plane, draw a straight line perpendicular to a straight line in space.

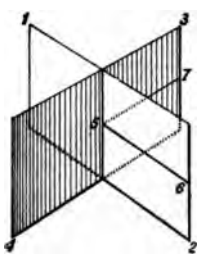
10. In any circle, inscribe three equal circles tangent to each other and the given circle.

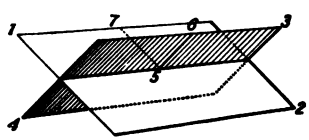
11. From a given point without the plane, inscribe a circle in the plane.

12. How long is the radius of a circle described by a line, 10 inches long, from a point 5 inches from the plane?

13. In a plane, to describe a circle having a radius of 6 inches, how long must the line be, if the point from which the circle is described is 8 inches above the plane?

SECTION II.—TWO PLANES.





DEFINITIONS.

436. A **dihedral angle** is the angle made by two planes that meet.

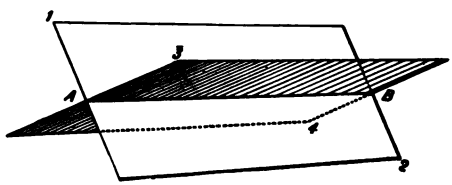
437. A dihedral angle is *measured* by the plane angle made by two straight lines, one in each plane, drawn perpendicular to the intersection at the same point.

The dihedral angle made by the planes ***1 2*** and ***3 4*** is measured by the plane angle ***7 5 6***, made by two lines, ***5 6*** and ***5 7***, drawn one in each plane perpendicular to the intersection. This may be proved as in the case of any two like quantities, whether commensurable or incommensurable. Illustrate with folded paper, a door, the covers of a book, or the lid of a box.

438. A **right dihedral angle** is a dihedral angle whose measure is a right angle.

PROPOSITION VI.

439. Theorem: ***The intersection of two planes is a straight line.***



Statement: Let the two planes *1 2* and *3 4* intersect. Prove that their intersection is a straight line.

Construction: Join *A* and *B*, two points in the intersection, by the straight line *A B*.

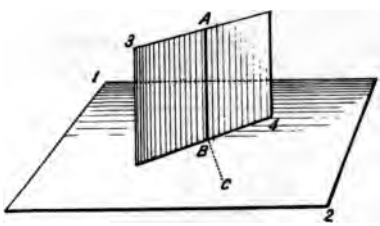
Demonstration: The straight line *A B* having two of its points, *A* and *B*, in the plane *1 2*, it lies wholly in the plane *1 2* (DEF.); and, for the same reason, *A B* lies wholly in the plane *3 4*. *A B*, therefore, lies wholly in both planes, and no point outside of *A B* is common to both planes; for, then, there would be two planes passing through the same straight line and the same point without that line; but there can be only one such plane, as the straight line and the point without it determine the position of the plane.

A B, the straight line, lies wholly in both planes, and contains all the points common to both planes; it is, therefore, the intersection of those planes.

Conclusion: *1 2* and *3 4* being any two planes, it follows, in general, that: The intersection of two planes is a straight line.

PROPOSITION VII.

440. Theorem: ***If a line is perpendicular to a plane, every plane passed through that line is perpendicular to the plane.***



Statement: If *A B* is perpendicular to the plane *1 2*, every plane, as *3 4*, passed through *A B*, is perpendicular to *1 2*.

Construction: In *1 2*, draw *B C* perpendicular to *B 4*.

Demonstration: Since *A B* is perpendicular to *1 2*, it is perpendicular to *B C*, a line in *1 2* passing through *B*. *A B C* is, then, a right angle; but *A B C* is, by definition, the measure of the dihedral angle made by the two planes. The one plane is, therefore, perpendicular to the other plane.

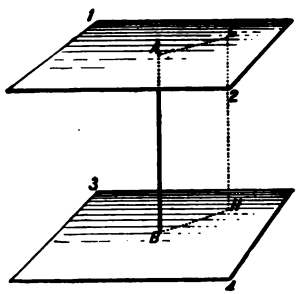
Conclusion: *1 2* being any plane, and *A B* being, etc.

Corollary I. A line in one of two perpendicular planes, perpendicular to the intersection, is perpendicular to the other plane.

Corollary II. A line perpendicular to one of two perpendicular planes, at a point of the intersection, lies in the other plane.

PROPOSITION VIII.

441. Theorem: ***If a straight line is perpendicular to one of two parallel planes, it is perpendicular, also, to the other plane.***



Statement: If *A B* is perpendicular to *3 4*, one of two parallel planes, *1 2* and *3 4*, it is, also, perpendicular to *1 2*.

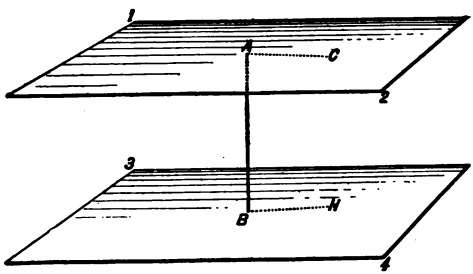
Construction: In *1 2*, draw any line, *A C*, and pass a plane through *A C* and *A B*, intersecting *3 4* in *B H*.

Demonstration: Since the planes *1 2* and *3 4* are parallel, they cannot meet. The line *A C*, therefore, cannot meet the line *B H*; but they are in the same plane, *A H*. *A C* and *B H* are, then, parallel. *A B* is, however, perpendicular to *B H*, by hypothesis; it is perpendicular to *A C*, then. (BOOK I., PROP. XIV.) *B A* being perpendicular to *A C*, any line in *1 2*, it is perpendicular to *1 2*.

Conclusion: *1 2* and *3 4* being any two parallel planes, and *A B* being any line perpendicular to one of them, it follows, in general, that: If a straight line, etc.

PROPOSITION IX.

442. Theorem: ***Planes perpendicular to the same straight line are parallel.***



Statement: If the planes *1 2* and *3 4* are perpendicular to the same straight line, *A B*, they are parallel.

Demonstration: The planes *1 2* and *3 4* are parallel; for, if they are not, they meet. If they meet, two straight lines, as *A C* and *B D*, can be drawn from the extremities of *A B*, one in each plane, to a point in the intersection. These two lines will thus meet; but, being each perpendicular to *A B*, by hypothesis, they are parallel and cannot meet. The planes, then, do not meet. They are parallel.

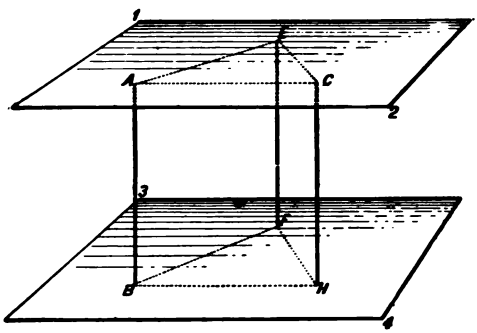
Conclusion: *1 2* and *3 4* being any two planes, etc.

Corollary I. A straight line and a plane perpendicular to the same line are parallel.

Corollary II. A line parallel to a line in a plane is parallel to the plane.

PROPOSITION X.

443. Theorem: ***Parallel lines between parallel planes are equal.***



Statement: If *A B*, *C H*, and *E F* are parallel lines intercepted between parallel planes, *1 2* and *3 4*, they are equal.

Construction: Draw *A C*, *C E*, and *A E*; also, *B H*, *H F*, and *B F*.

Demonstration: *A C* and *B H* are parallel, as they lie in the same plane, *A H*; and, being in parallel planes, they cannot meet. In the same manner, *A E* and *B F* are proved parallel; also, *C E* and *H F*. The figures *A H*, *C F*, and *A F* are, therefore, parallelograms, and the opposite sides, *A B*, *C H*, and *E F*, are equal. (BOOK III., PROP. I.)

Conclusion: *A B*, *C H*, and *E F* being any parallel lines, etc.

Corollary I. Parallel planes are everywhere equally distant.

Corollary II. The intersections of a plane with two parallel planes are parallel straight lines.

Corollary III. Angles lying in different planes and having sides respectively parallel are equal.

Corollary IV. Lines parallel to the same line are parallel to each other.

MISCELLANEOUS EXERCISES.

444. 1. If a straight line intersects two parallel planes, it makes equal angles with them.

2. Two planes which are parallel to the same straight line are either parallel to each other or their intersection is parallel to this line.

3. A line parallel to each of two intersecting planes is parallel to their intersection.

4. Two planes which are parallel to two planes which intersect each other will also intersect, and the line of intersection of the first planes will be parallel to the line of intersection of the second.

5. Find the locus of all the points at a given distance from a given plane.

6. Through a given point, draw a straight line parallel to two given planes.

7. The angles which two parallel lines, oblique to a plane, make with that plane are equal.

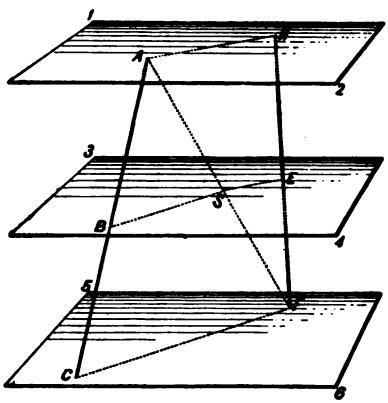
8. Find the locus of all the points, any one of which is equally distant from two given points.

9. Any point on the plane bisector of a dihedral angle is equally distant from the faces of the angle ; any point not in this plane is unequally distant from the faces.

SECTION III.—SEVERAL PLANES.

PROPOSITION XI.

445. Theorem: ***Lines intersecting parallel planes are divided into proportional segments.***



Statement: Let *A B C* and *H E F* be two straight lines cut by parallel planes *1 2*, *3 4*, and *5 6*. The segments are proportional; that is, *A B : H E : : B C : E F.*

Construction: Draw *A F*, *A H*, *B S*, *S E*, and *C F*.

Demonstration: *B S* and *C F* are parallel (PROP. X., COR. II.), and, also, *E S* and *A H*; therefore (BOOK VII., PROP. VI.),

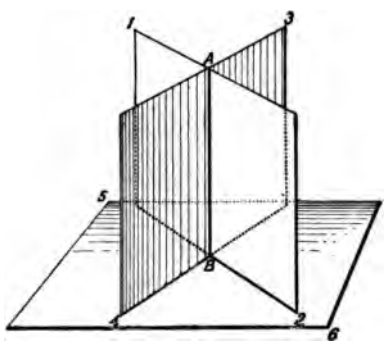
A B : A S : : B C : S F, and, *H E : A S : : E F : S F*;

therefore, *A B : H E : : B C : E F.* (BOOK V., PROP. XIII.)

Conclusion: *A C* and *H F* being any two lines, etc.

PROPOSITION XII.

446. Theorem: ***If two intersecting planes are perpendicular to a third plane, their intersection is perpendicular to that plane.***



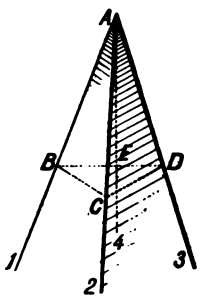
Statement: Let *1 2* and *3 4* be two intersecting planes perpendicular to *5 6*. Prove their intersection perpendicular to *5 6*.

Demonstration: The plane *1 2* intersects *5 6* in *B 2*; and, *3 4* intersects *5 6* in *B 4*. (PROP. VI.) At *B*, erect a perpendicular to *5 6*; it will lie in *1 2* and in *3 4*. (PROP. VII., COR. II.) *A B*, being in *1 2* and in *3 4*, is, then, their intersection (PROP. VI.), and it is perpendicular to *5 6*.

Conclusion: *1 2* and *3 4* being any two intersecting planes, perpendicular to a third plane, it follows, in general, that: If two intersecting planes are perpendicular to a third plane, their intersection is perpendicular to that plane.

Proposition XIII.

447. Theorem: ***The sum of any two of the three plane angles formed by the edges of a trihedral angle is greater than the third angle.***



Statement: Let *A* be any trihedral angle. *1 A 2* and *2 A 3*, any two of the three plane angles formed by the edges of the trihedral angle, are greater than *1 A 3*, the third plane angle.

Construction: If any two of the plane angles are equal, the theorem is obviously true. Conceiving *1 A 3* to be greater than *1 A 2*, take a part, *1 A 4*, equal to *1 A 2*. Draw *B E D*, any line in the plane *1 A 3*. Take *A C*, equal to *A E*, and draw *B C* and *C D*.

Demonstration: The triangles *B A C* and *B A E* are equal. (Book II., Prop. I.) $BC + CD > BD$;

subtracting, $BC = BE$, and $CD > ED$.

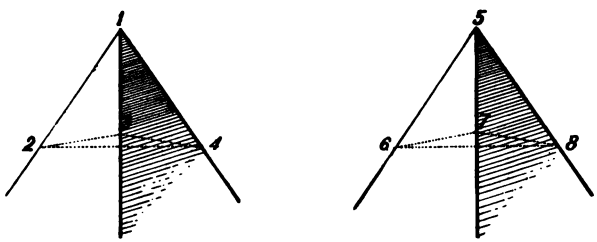
In triangles *C A D* and *D A E*, then, the angle $CAD > DAE$ (Book II., Prop. XV.); and, adding $BAC = BAE$,

$$BAC + CAD > BAE + EAD, \text{ or } BAD.$$

Conclusion: *A* being any trihedral angle, etc.

PROPOSITION XIV.

448. Theorem: ***If the plane angles of two trihedral angles are equal, each to each, the planes of the equal angles are equally inclined to each other.***



Statement: Let *1–2 3 4* and *5–6 7 8* be two trihedral angles with equal plane angles. Prove the dihedral angles equal.

Construction: Take *1 3*, equal to *5 7*. Draw *3 2*, *3 4*, *7 6*, and *7 8*, perpendicular to *1 3* and *5 7*, in respective planes.

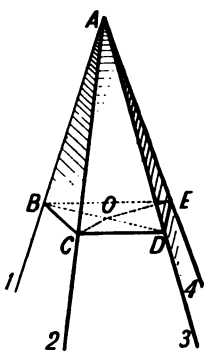
Demonstration: The triangles *1 2 3* and *5 6 7* are equal (BOOK II., PROP. II.); as, also, *1 3 4* and *7 5 8*. *2 3* and *6 7* are, therefore, equal; also, *3 4* and *7 8*. The triangles *1 2 4* and *5 6 8* are equal. (BOOK II., PROP. I.) *2 4* and *6 8* are, therefore, equal. The two triangles *2 3 4* and *6 7 8* are, then, equal. (BOOK II., PROP. III.) The plane angles *2 3 4* and *6 7 8*, measuring the corresponding dihedral angles, are thus proved equal. In the same manner, the other dihedral angles are proved equal.

Conclusion: *1–2 3 4* and *5–6 7 8* being any two trihedral, etc.

Corollary. Trihedral angles whose plane angles are equal, each to each, may be made to coincide.

PROPOSITION XV.

449. Theorem : ***The sum of the plane angles made by the edges of any convex polyhedral angle is less than four right angles.***



Statement : Let *A–1 2 3 4* be any convex polyhedral angle. Prove that the sum of the plane angles formed by its edges is less than four right angles.

Construction : Intersect the faces of the polyhedral angle by a plane, forming the polygon *B C D E*. From any point in the polygon, draw lines to its vertices ; as, *O B, O C*, etc.

Demonstration : There are two sets of triangles, one set having vertices at *A ;* and the other set, at *O*. The number of triangles in the two sets is the same, as for each side of the polygon there is a triangle in each set. The sum of the plane angles in the two sets is, therefore, the same. The angles at the base of the oblique set are, however, greater than the angles at the base of the horizontal set. (PROP. XIII.) The plane angles at the vertex *A* are, therefore, less than the angles at *O ;* that is, the plane angles at *A* are less than four right angles.

Conclusion : *A–1 2 3 4* being any polyhedral angle, etc.

MISCELLANEOUS EXERCISES.

450. 1. If two parallel lines intersect a plane, they make equal angles with it.

2. The three planes which bisect the dihedral angles of a trihedral angle meet in the same straight line.

3. Find the locus of all the points, any one of which is equally distant from the three edges of a trihedral angle.

4. Through a given point, draw a plane perpendicular to two given planes.

5. Find the locus of all the points, any one of which is equidistant from two given planes.

6. A straight line and plane which are perpendicular to the same plane are parallel.

7. Through a given point, draw a plane parallel to two given straight lines.

8. Through a given straight line, draw a plane which shall be parallel to a given straight line.

9. All straight lines, perpendicular at a given point to a given straight line, lie in a plane perpendicular to the given line at the given point.

10. From a given point without a straight line, one, and but one plane can be drawn to that line.

11. If a straight line be drawn from the foot of a perpendicular to a plane perpendicular to any line in the plane, any line drawn from the latter point of intersection to any point in the perpendicular to the plane will be perpendicular to the line lying in the plane.

12. A straight line in one of two perpendicular planes, drawn perpendicular to the intersection of the two planes, is perpendicular to the other plane.

13. A straight line in one of two perpendicular planes, perpendicular to the intersection, lies in the other plane.

14. Find the locus of the points each of which is equally distant from all the points in the circumference of any given circle in a given plane.

15. Find the locus of the points in space equally distant from any two given points.

16. Any plane passed through a line perpendicular to a plane is itself perpendicular to the given plane.

17. A plane perpendicular to each of two other planes is perpendicular to the intersection of these planes.

18. A plane passed through one, and only one, of two parallel straight lines is parallel to the other line.

19. Through a given straight line pass a plane parallel to any other straight line.

20. Through any given straight line without a plane, one, and but one plane can be perpendicular to a given plane.

21. Two lines intersected by three planes are 12 and 20 inches long. If the segments of the former are 9 and 3, what are the segments of the latter?

22. The sum of the dihedral angles formed by one plane meeting another plane is equal to two right angles.

23. From a point without a plane, drop a perpendicular.

24. From a point on a plane, erect a perpendicular.

25. If two planes intersect, their vertical angles are equal.

26. If one plane intersect obliquely two parallel planes, the four acute dihedral angles thus formed are equal, and, also, the four obtuse dihedral angles.

27. The two interior angles on the same side are equal to two right angles.

28. One perpendicular, and but one, can be drawn to two straight lines not in the same plane.

BOOK IX.—SOLIDS WITH PLANE SURFACES.

SYNOPSIS.

SECTION I.—SECTIONS.

451. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. I. Parallel cross sections of a prism are equal.

PROP. II. Parallel cross sections of a pyramid are similar polygons.

PROP. III. The areas of parallel cross sections of a pyramid are proportional to the squares of their distances from the apex of the pyramid.

3. MISCELLANEOUS EXERCISES.

SECTION II.—SURFACES.

452. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. IV. The lateral surface of any prism is equal to the perimeter of any right section multiplied by any one of the lateral edges.

PROP. V. The lateral surface of a regular pyramid is equal to half the product of the perimeter of the base and the slant height.

PROP. VI. The lateral surface of the frustum of a regular pyramid is equal to the product of the sum of the perimeters of the bases by half of the slant height.

3. MISCELLANEOUS EXERCISES.

SECTION III.—EQUIVALENT SOLIDS.

453. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. VII. If the three faces forming any trihedral angle of one prism are equal, each to each, to the three faces forming any trihedral angle of another prism, and are similarly placed, the prisms are equal.

PROP. VIII. Oblique parallelopipeds are equivalent to right rectangular parallelopipeds with equivalent bases and the same altitudes.

PROP. IX. Pyramids with equivalent bases and the same altitude are equivalent.

3. MISCELLANEOUS EXERCISES.

SECTION IV.—PROPORTIONAL SOLIDS.

454. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. X. Parallelopipeds with equivalent bases are proportional to their altitudes.

PROP. XI. Parallelopipeds with equal altitudes are proportional to their bases.

PROP. XII. Parallelopipeds are proportional to the products of their three dimensions.

3. MISCELLANEOUS EXERCISES.

SECTION V.—VOLUMES.

455. 1. DEFINITIONS.

2. PROPOSITIONS.

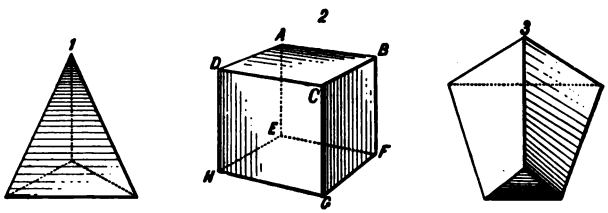
PROP. XIII. The volume of a prism is equal to the product of its three dimensions.

PROP. XIV. The volume of a triangular pyramid is equal to one-third of the product of its base and altitude.

PROP. XV. The volume of any frustum of a triangular pyramid is equal to the volume of three pyramids having for their altitude the altitude of the frustum, and for their bases, the bases of the frustum and a mean proportional between the two bases of the frustum.

3. MISCELLANEOUS EXERCISES.

SECTION I.—SECTIONS.



DEFINITIONS.

456. A **solid** is a portion of space bounded by one or more surfaces.

In *Geometry*, a solid is the space, and not the material that occupies the space. **Volume** is the word which refers to the *quantity* of space in a solid.

457. A plane (or square) solid is a solid bounded by plane surfaces; and a curved (or round) solid, a solid bounded by surfaces all or some of which are curved.

458. If the solid is inclosed by *four planes*, it is called a **tetrahedron.**

Figure *1* represents a tetrahedron. Four is the least number of plane surfaces that can inclose a solid.

459. If the solid is inclosed by *five planes*, it is a **pentahedron**; if by *six planes*, a **hexahedron.**

The name of the solid is taken from the number of planes in its surface, Greek numerals, *tetra*, *penta*, *hexa*, *hepta*, *octa*, *nona*, *deka*, etc., being prefixed to *edron*, or *hedron*. All *plane solids* are called *polyhedrons*, because they have *many* faces.

NOTE.—Point to a number of solids, and name them with reference to the number of faces. What do figures **2** and **3** represent?

460. A **prism** is a plane solid two of whose faces are polygons, with corresponding sides equal and parallel. All of the other faces are parallelograms. The two polygons with sides equal and parallel are generally called the *bases*.

The lines in which the planes intersect are called the **edges**. The points in which the edges intersect are called the **vertices**.

461. Prisms with triangular bases are called **triangular prisms**; with quadrilateral bases, **quadrangular prisms**, etc., prisms being named from their bases.

Point to a number of prisms. Represent some on the blackboard.* Name them by letters. State the kind of prism.

462. A **parallelopiped** is a plane solid all of whose faces are parallelograms.

463. Parallelopipeds, and all other prisms, are said to be **right parallelopipeds** or **right prisms** when the lateral edges are perpendicular to the bases, and they are said to be **oblique parallelopipeds** or **oblique prisms** when the lateral edges are not perpendicular to the bases. In right prisms the lateral faces are all rectangles.

464. A **regular prism** is a right prism whose base is a regular polygon.

465. A **rectangular parallelopiped** is a parallelopiped whose bases are rectangles. All of the faces of a right rectangular parallelopiped are rectangles.

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* Draw the upper base first; then draw the lateral edges, all of the same length, allowing for perspective, if desired; and then draw the lower base by uniting the lower extremities of the lateral edges. The lines and parts of lines covered by planes should be represented by dotted lines. In naming the prism by letter, one of the bases and one point in the other base are generally named.

466. A **cube** is a plane solid bounded by six equal squares.

467. A **pyramid** is a *polyhedron*, or plane solid, having a polygon for its base, and triangles with a common vertex for its lateral surface.

The common vertex of the triangles is called the **apex** of the pyramid.

468. A pyramid with a triangle for its base is a **triangular pyramid**; with a square for its base, a **square pyramid**; with a rectangle for its base, a **rectangular pyramid**, etc., pyramids being named from their bases.

469. A **regular pyramid** is a pyramid whose base is a regular polygon, and whose apex is perpendicularly above the center of the base.

470. The **lateral surface** of a pyramid is the sum of all of the faces except the base. The **lateral edges** are the intersection of the triangles forming the lateral surface.

471. The **altitude of a prism** is the perpendicular distance between the bases.

472. The **altitude of a pyramid** is the perpendicular distance between the apex and the base.

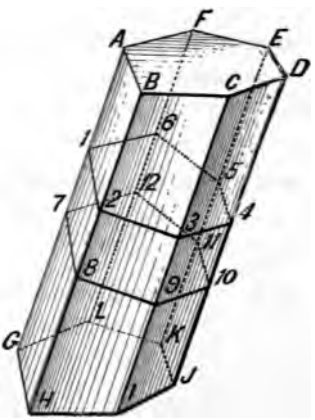
473. The intersection of a plane with a solid is called a **section.**

474. The **right section** of a prism is the section made by a plane perpendicular to the lateral edges of the prism.

A section that cuts the lateral edges is called a **cross section.** A section that cuts the bases is called a **longitudinal section.**

PROPOSITION I.

475. Theorem: ***Parallel cross sections of a prism are equal.***



Statement: Let *A B C D E F–G* be any prism. If *1 4* and *7 10* are parallel cross sections, they are equal polygons.

Demonstration: The corresponding sides of the sections, as *1 2* and *7 8*, are parallel, being intersections of two parallel planes with a third plane. (BOOK VIII., PROP. X., COR. II.) The lines *1 7*, *2 8*, *3 9*, etc., are parallel, because they are parts of the lateral edges. The figures *1 2 7 8*, *2 3 9 8*, *3 4 10 9*, etc., are, then, parallelograms.

The corresponding sides of the cross sections are, then, equal; as, *1 2* to *7 8*, *2 3* to *8 9*, etc. (BOOK III., PROP. I.)

The lines *1 2* and *7 8* having been proved parallel, as, also, *2 3* and *8 9*, the angle *1 2 3* is equal to the angle *7 8 9*. (BOOK VIII., PROP. X, COR. III.) In similar manner, the other corresponding angles may be proved equal; as, *3* to *9*, etc.

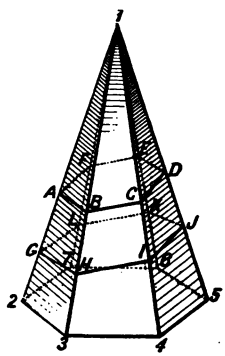
Conclusion: As *A J* is any prism, and as *1 4* and *7 10*, etc.

Corollary I. A section parallel to the base equals the base.

Corollary II. Opposite faces of parallelopipeds are equal and parallel parallelograms.

PROPOSITION II.

476. Theorem: ***Parallel cross sections of a pyramid are similar polygons.***



Statement: Let *A B C D E F* and *G H I J K L* be parallel sections of the pyramid *1–2 3 4 5 6 7*. They are similar polygons.

Demonstration: The corresponding sides, as *A B* and *G H*, are parallel. (BOOK VIII., PROP. X., COR. II.) The corresponding angles, as *A B C* and *G H I*, are, therefore, equal. (BOOK VIII., PROP. X., COR. III.)

The sides of the triangles *1 A B* and *1 G H* are parallel, and the triangles are, then, similar (BOOK VII., PROP. IV.); hence, *A B : G H : : 1 B : 1 H*; similarly, *B C : H I : : 1 B : 1 H*; and, therefore, *A B : G H : : B C : H I*. (BOOK V., PROP. XI.)

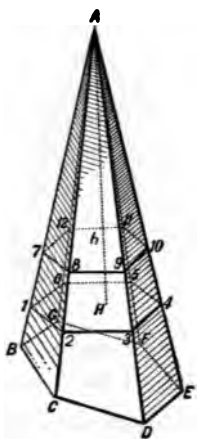
In similar manner, the other sides are proved proportional.

Conclusion: *1–2 3 4 5 6 7* being any pyramid, etc.

Corollary I. Sections parallel to the base are similar to the base.

PROPOSITION III.

477. Theorem: ***The areas of parallel cross sections of a pyramid are proportional to the squares of their distances from the apex of the pyramid.***



Statement: Let *A–B C D E F G* be any pyramid. If *1 2 3 4 5 6* and *7 8 9 10 11 12* are any parallel cross sections, and *A H* and *A h* are their perpendicular distances from the apex, the areas of *1 4* and *7 10* are proportional to $A\,H^2$ and $A\,h^2$; that is,

$$1\,4 : 7\,10 :: A\,H^2 : A\,h^2.$$

Demonstration: If *A H* is perpendicular to *1 4*, it is also perpendicular to *7 10*. (BOOK VIII., PROP. VIII.)

The cross sections being similar polygons (PROP. II.), they are proportional to the squares of their corresponding sides (BOOK VII., PROP. XXI.); thus,

$$14 : 710 :: 12^2 : 78^2. \quad \text{(A)}$$

As $A78$ and $A12$ are similar triangles (Book VIII., Prop. X., Cor. II., and Book VII., Prop. IV.), $12 : 78 :: A2 : A8$;

but, $A2 : A8 :: AH : Ah$ (Book VIII., Prop. XI.);

therefore, $12 : 78 :: AH : Ah$ (Book V., Prop. XI.);

and, $12^2 : 78^2 :: AH^2 : Ah^2$. (B) (Book V., Prop. XIX.)

A comp1et in each of the two proportions being the same,

$$14 : 710 :: AH^2 : Ah^2 \text{ (Book V., Prop. XI.)};$$

that is, the area of the cross section is proportional to the square of its distance from the apex.

Conclusion: $A\text{–}BCDEFG$ being any pyramid, etc.

Corollary I. If a cross section is parallel to the base, its area is to the area of the base, as the square of its distance from the apex is to the square of the altitude of the pyramid.

Corollary II. If two or more pyramids have equal altitudes and equivalent bases, their cross sections, parallel to the bases, at equal distances from the apices, are equivalent.

MISCELLANEOUS EXERCISES.

478. 1. If the four diagonals of a quadrangular prism pass through a common point, the prism is a parallelopiped.

2. Every section of a prism by a plane parallel to any lateral edge is a parallelogram.

3. Any straight line drawn through the center of a parallelopiped, terminating in a pair of opposite faces, is bisected at that point.

4. The diagonals of a right rectangular parallelopiped are equal.

5. The lateral edges of a parallelopiped are all parallel.

6. In any parallelopiped, the four edges meeting any face are parallel to one another.

7. The diagonals of a parallelopiped bisect each other.

8. The square of any diagonal of a right rectangular parallelopiped is equal to the sum of the squares of the three edges meeting at any vertex.

9. Any lateral edge of a right prism is equal to the altitude.

10. The sum of the squares of the four diagonals of a parallelopiped equals the sum of the squares of all the edges.

11. The lateral faces of right prisms are rectangles.

12. The lateral faces of a regular prism are rectangles.

13. What is the length of any diagonal of a right rectangular parallelopiped whose edges are 3, 4, and 5 ?

14. Any straight line drawn through the center, and terminating in opposite faces of a parallelopiped is bisected at the center.

15. If a pyramid is intersected by a plane parallel to the base, the lateral edges and altitude are cut proportionally.

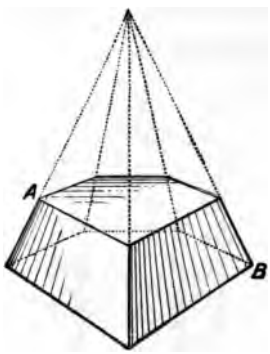
16. Find the lateral edge of a regular triangular pyramid, each side of the base being 4, and the altitude, 3.

17. Find the locus of the points equally distant from the edges of a trihedral angle.

18. A line parallel to two intersecting planes is parallel to their intersection.

19. The plane angle formed by two perpendiculars from any point in a dihedral angle is the supplement of the dihedral angle.

SECTION II.—SURFACES.



DEFINITIONS.

479. A **frustum** of a pyramid is a part of a pyramid included between the base and a plane parallel to the base; as *A B*, in the diagram.

480. The base of the pyramid and the face parallel to it are called the **bases** of the frustum.

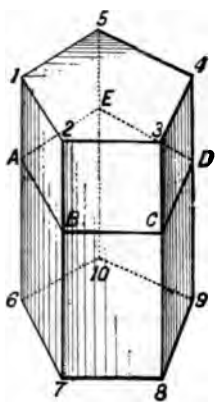
The trapezoids are the *lateral faces*, and their sum is the lateral surface of the frustum.

481. The **slant height** of a **regular pyramid** is the perpendicular distance between the apex and any one of the edges at the base of the pyramid.

482. The **slant height** of a **frustum** of a **regular pyramid** is the perpendicular distance between the parallel edges of any lateral face.

PROPOSITION IV.

483. Theorem: ***The lateral surface of any prism is equal to the perimeter of any right section multiplied by any one of the lateral edges.***



Statement: Let *1-8* be any prism, and let ***A B C D E*** be any right cross section. The lateral surface of the prism equals the broken line ***A B C D E*** multiplied by any lateral edge; as ***1 6***.

Demonstration: By definition, the lateral faces are all parallelograms. All of the lateral edges are, then, parallel (BOOK VIII., PROP. X., COR. IV.); and they are equal. (BOOK VIII., PROP. X.)

As the lateral edges are perpendicular to a right section, they are perpendicular to ***A B***, ***B C***, ***C D***, ***D E***, and ***E A***, which are lines in this plane passing through their feet.

The lines ***A B***, ***B C***, etc., may represent the altitudes of the parallelograms of which the lateral edges are the bases. The

area of any lateral face is, then, equal to a lateral edge multiplied by its intersection with the right section and the area of all the lateral faces, that is, the area of the lateral surface is equal to the lateral edge multiplied by the sum of these intersections, that is, by the perimeter of the right section. (BOOK VI., PROP. XIV.)

Conclusion: *1–8* being any prism, and ***A B C D E*** being, etc.

Corollary I. The lateral surface of a right prism is equal to the product of the perimeter of its base by a lateral edge.

Corollary II. The entire surface equals the lateral surface together with the area of the bases.

Exercises. 1. Find the lateral area of a regular triangular prism whose altitude is 10, each side of the base being 6.

2. Find the entire surface of the solid in the preceding exercise.

3. What is the lateral surface of a regular square prism whose altitude is 12, each side of the base being 6 ?

4. Find the entire surface of the solid in the preceding exercise.

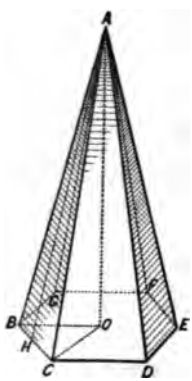
5. Find the lateral surface of a regular hexagonal prism whose altitude is 10, each side of the base being 5.

6. What is the entire surface of the solid in the preceding exercise ?

7. The corresponding edges of two regular square prisms are as 4 : 5. If the surface of the former is 40 square inches, what is the surface of the latter ?

PROPOSITION V.

484. Theorem: ***The lateral surface of a regular pyramid is equal to the perimeter of its base multiplied by half of the slant height.***



Statement: Let *A-B C D E F G* be a regular pyramid, and let *A H* be its slant height. The lateral surface is equal to *B C D E F G* multiplied by one-half of *A H*.

Construction: Draw *A O*, the altitude of the regular pyramid, and draw from *O* lines, as *O B*, *O C*, etc., to the vertices of the base.

Demonstration: By the definitions of a regular pyramid and of a regular polygon, *A O B*, *A O C*, etc., are right-angled triangles, with two sides and the included angles equal. The lateral edges *A B*, *A C*, etc., are, then, equal to one another. (BOOK II., PROP. I) The lateral faces, having three sides of the one equal to the three sides of the other, each to each,

are equal in all their parts. (BOOK II., PROP. III.) The straight lines drawn from *A*, the apex, perpendicular to the bases of the lateral faces, that is, the slant heights, as *AH*, are all equal. (BOOK II., PROP. V.)

The area of any one of the lateral faces is equal to the product of its base by one-half of the slant height. (BOOK VI., PROP. IX.) The area of the lateral surface, or the sum of the lateral faces, is equal to the sum of all of the bases of the lateral faces, or the perimeter of the base of the pyramid, multiplied by half of the slant height. (AXIOM II.)

Conclusion. *A–B C D E F G* being any regular pyramid, etc.

Corollary I. The lateral surface of any pyramid may be found by adding together the areas of the lateral faces.

Corollary II. The entire surface of a pyramid is equal to the lateral faces together with the base.

Exercises. 1. Find the lateral surface of a regular triangular pyramid whose altitude is 10, each side of the base being 6.

2. Find the entire surface of the solid in preceding exercise.

3. What is the lateral surface of a regular square pyramid whose altitude is 12, each side of the base being 6?

4. Find the entire surface of the solid in the preceding exercise.

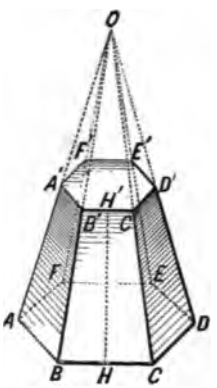
5. What is the lateral surface of a regular hexagonal pyramid whose altitude is 10, each side of the base being 5?

6. Find the entire surface of the solid in preceding exercise.

7. The corresponding edges of two regular square pyramids are as 4 : 5. The lateral surface of the former is 50 square inches. What is the lateral surface of the latter?

PROPOSITION VI.

485. Theorem: ***The lateral surface of the frustum of a regular pyramid is equal to the product of the sum of the perimeters of the bases by half of the slant height.***



Statement: Let $A'D$ be the frustum of a regular pyramid. The lateral surface is equal to the sum of the perimeters, $ABCDEF$ and $A'B'C'D'E'F'$, multiplied by half of HH', the slant height.

Construction: Complete the regular pyramid.

Demonstration: As $A'B'C'D'E'F'$ is a section parallel to the base, it is a regular polygon (PROP. II.), and $O\text{–}A'B'C'D'E'F'$ is, then, a regular pyramid.

As in PROP. V., the lateral faces of each of the two pyramids in the diagram are equal to one another.

Subtracting the faces of the small pyramid from the corre-

sponding faces of the large pyramid, there remain the trapezoids $A'B$, $B'C$, $C'D$, etc, equal in all their parts. (AXIOM IV.) The slant height of the small pyramid, subtracted from that of the large pyramid, gives the slant height of the frustum.

The area of any lateral face of the frustum is equal to the sum of its parallel sides by one-half of the slant height; that is, one-half of the perpendicular distance between the parallel sides. (BOOK VI., PROP. XV.) The area of all of the lateral faces, that is, the lateral surfaces of the frustum, is equal to the sum of all the parallel sides, or the perimeters of the two bases, multiplied by one-half of the slant height. (AXIOM II.)

Conclusion: $A'D$ being any frustum of any regular, etc.

Corollary I. The lateral surface of any frustum of any pyramid may be found by adding together the sum of the areas of the lateral faces.

Corollary II. The entire surface of any frustum may be found by adding to the lateral surface the areas of the bases.

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486. Exercises. 1. Prove that the lateral surface of a pyramid is greater than its base.

2. The corresponding edges of two similar tetrahedrons are as 3 : 4. What is the ratio of the surfaces?

3. Find the lateral surface of a frustum of a regular hexagonal pyramid, the sides of the bases being 4 and 5, and the altitude, 6.

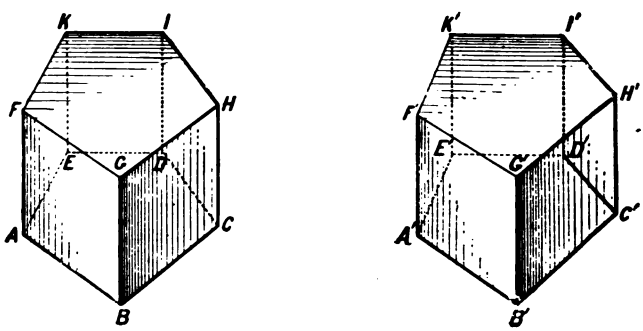
4. What is the entire surface of a triangular pyramid each of whose edges is 5?

SECTION III.—EQUIVALENT SOLIDS.

487. Solids are *equivalent* when they are equal in volume. They are equal, when they are equal in all their parts.

PROPOSITION VII.

Theorem: ***If the three faces forming any trihedral angle of one prism are equal, each to each, to the three faces forming any trihedral angle of another prism, and are similarly placed, the prisms are equal.***



Statement: Let AH and $A'H'$ be two prisms, having the three faces BF, BH, and BD forming the trihedral angle B, equal, respectively, to three faces, $B'F'$, $B'H'$, and $B'D'$, forming the trihedral angle B'; and, let the equal faces be similarly placed. The prisms AH and $A'H'$ are equal in all their parts.

Demonstration: Suppose that the prism $A'H'$ be applied to the prism AH, so that the equal bases, BD and $B'D'$, shall coincide.

The corresponding plane angles at B and B' being equal, by hypothesis, as they are the corresponding parts of equal faces, the trihedral angles B and B' coincide, and the lateral edge $B'G'$ must take the same direction as the lateral edge BG. (BOOK VIII., PROP. XIV., COR.)

All the other corresponding lateral edges must coincide in direction, for their lower extremities coincide, and they are parallel to the lines $B'G'$ and BG, which have already been proved to coincide in direction. (BOOK I., PROP. XIX., COR.)

The upper extremities of the corresponding lateral edges must coincide, for the lateral edges are all equal (BOOK VIII., PROP. X.), being equal to BG and $B'G'$, which are equal.

The vertices of the upper bases thus coinciding, the sides of these bases must coincide, as between any two points only one straight line can be drawn. The prisms have thus been proved to coincide, throughout, and they are, then, equal.

Conclusion: AH and $A'H'$ being any two prisms, etc.

Corollary I. A plane passed through the diagonally opposite edges of a parallelopiped divides the parallelopiped into equal triangular prisms.

Corollary II. Truncated prisms are equal, if they have three faces of any trihedral angle of one similarly placed and equal, each to each, to three faces of a trihedral angle of the other.

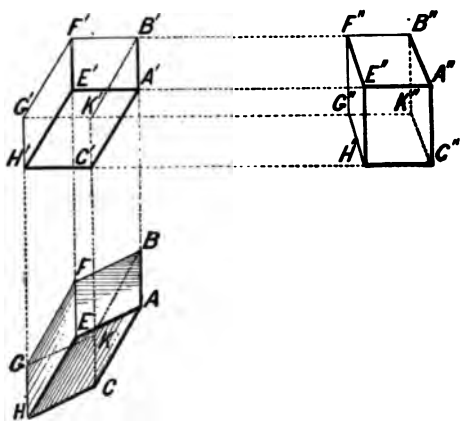
A truncated prism is a part of a prism included between a base and a section not parallel to the base.

Corollary III. An oblique prism is equivalent to a right prism with the same lateral edge, and a base which is a right section of the oblique prism.

Equal subtractions of two equal truncated prisms.

PROPOSITION VIII.

488. Theorem: ***Oblique parallelopipeds are equivalent to right rectangular parallelopipeds with equivalent bases and the same altitude.***



Statement: Let *A G* be any oblique parallelopiped. It will be equivalent to a right rectangular parallelopiped with equivalent base and the same altitude.

Construction: Produce *A B*, *C K*, *E F*, and *H G*, and construct between them *A' G'*, an oblique parallelopiped with equal rectangular base, the lines *H G* and *H' G'* being equal. Produce *E' A'*, *F' B'*, *H' C'*, and *G' K'*, and construct between them a right rectangular parallelopiped, *A'' G''*, with an equivalent rectangular base, *K''*, the lines *H' C'* and *H'' C''* being equal.

Demonstration: The three parallelopipeds have, by construction, equivalent bases, and the same altitude.

The solids AH' and BG', truncated prisms, may be proved to be equal by proving that the faces about the trihedral angles A and A' are equal. (PROP. VII., COR. II.) The face AC' is equal to the face BK', as all the sides and angles are equal, each to each. The side AC, of the one, equals the side BK, of the other, as they are opposite sides of the parallelogram AK; $A'C'$, of the one, equals $B'K'$, of the other, as they are opposite sides of $A'K'$; AA', of the one, equals BB', of the other, as each equals the line BA', together with AB or $A'B'$, these latter lines being equal, by construction. In the same way, CC', of the one, is shown to be equal to KK', of the other; and the corresponding angles are all equal, because they are included between parallel lines. The face AC' is, then, equal in all its parts to the face BK'. In the same way, the face AE' may be shown to be equal to the face BF'; while the face AH is equal to the face BG, being opposite faces of the parallelopiped AG.

The three faces about the trihedral angle A, of the truncated prism AH', have thus been proved equal, each to each, to the three faces about the trihedral angle B, of the truncated prism BG'. AH' and BG' are, then, equal. (PROP. VII., COR. II.) Subtracting the common part BH' from AH' and BG', there remain the equivalent parallelopipeds AG and $A'G'$. (AXIOM IV.) The two parallelopipeds AG and $A'G'$ have thus been proved to be equivalent.

In similar manner, the two parallelopipeds $A'G'$ and $A''G''$ may be proved to be equivalent.

The parallelopipeds AG and $A''G''$ being each equivalent to the parallelopiped $A'G'$, they are equivalent to each other.

Conclusion: The parallelopiped AG being any oblique parallelopiped, etc.

Corollary I. Parallelopipeds having equivalent bases and the same altitude are equivalent.

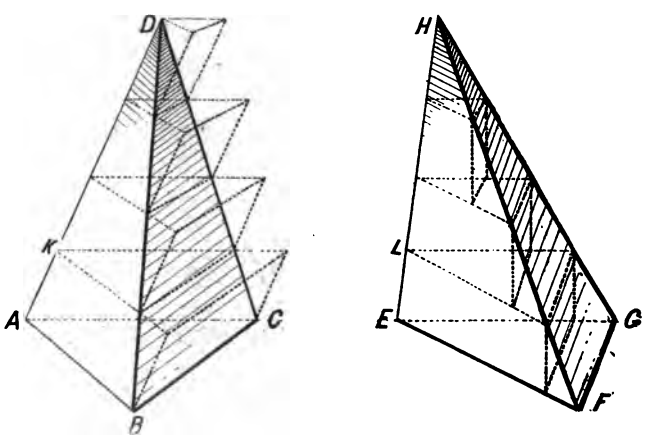
Corollary II. Triangular prisms with equivalent bases and the same altitude are equivalent. (PROP. VII., COR.)

Corollary III. Prisms having equivalent bases and the same altitude are equivalent.

Corollary IV. Any prism is equivalent to a right prism with a cross section for its base and a lateral edge for its altitude.

PROPOSITION IX.

489. Theorem: ***Pyramids with equivalent bases and the same altitude are equivalent.***



Statement: Let *D–A B C* and *H–E F G* be any two triangular pyramids, with equivalent bases, *B C D* and *E F G*, and the same altitude. Prove the pyramids equivalent.

Construction: Conceiving the common altitude to be divided into any number of equal parts, through the points of division pass planes parallel to the bases. On ***A B C***, and the cross sections in ***D–A B C***, construct prisms, with lateral edges equal and parallel to ***A K***. Inside of ***H–E F G***, on the cross sections as upper bases, construct prisms, with lateral edges equal and parallel to ***E L***.

Demonstration: The corresponding cross sections are equivalent (PROP. III., COR.) Each prism in ***H–E F G*** is, then, equivalent to the next higher prism in ***D–A B C***. (PROP. VIII., COR. III.) The difference between the two sets of prisms is the prism ***K–A B C***.

As the one set of prisms is greater than ***D–A B C***, and the other, less than ***H–E F G***, the difference between the pyramids is less than the difference between the sets of prisms. The difference between the pyramids is, then, less than ***K–A B C***.

By indefinitely decreasing the altitude of the prisms, or increasing the number of sections and prisms, the difference between the pyramids may become less than any assignable quantity ; the volumes of the sets of prisms finally becoming the volumes of the pyramids, without difference. The volumes of the pyramids are, then, equivalent.

Conclusion : ***A–B C D*** and ***H–E F G*** being any, etc.

Exercises. 1. Find the edges and the lateral surface of the frustum of a regular square pyramid, the sides of the bases being 10 and 5, and the altitude being 8.

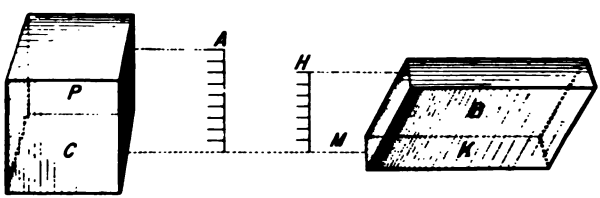
2. What is the entire surface of a triangular pyramid, each edge being 10?

3. Find the altitude of the solid in the preceding exercise.

SECTION IV.—PROPORTIONAL SOLIDS.

PROPOSITION X.

490. Theorem: ***Parallelopipeds with equivalent bases are proportional to their altitudes.***



Statement: Let the parallelopipeds *P* and *B* have equivalent bases, *C* and *D*, and let *A* and *H* be their respective altitudes. Prove that *P* and *B* are proportional to *A* and *H*.

I. Commensurable altitudes.

Demonstration: Suppose the altitudes *A* and *H* contain their common measure, *M*, respectively nine and seven times; then,

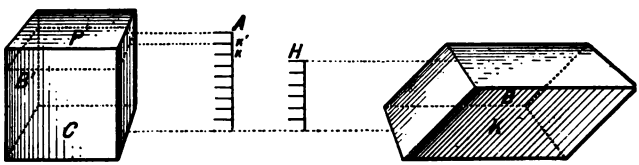
$$A : H :: 9 : 7.$$

Through the points of division of the altitude, pass planes parallel to the bases. As the cross sections thus made are equal and parallel to their bases (PROP. I.), they divide *P* into nine and *B* into seven equal parallelopipeds. (PROP. VIII., COR. I.) As *P* contains nine of the equal parallelopipeds, and *B*, seven, it follows, that,

$$P : B :: 9 : 7;$$ and, therefore

(BOOK V., PROP. XI.), $P : B :: A : H.$ *Q. E. D.*

II. Incommensurable altitudes.



Demonstration: If it is not true that,

$$P : B :: A : H,$$

imagine that, $P : B :: A : K,$ (A)

in which K is any altitude greater than H.

Divide A into equal parts, each of which is less than AK. At least one point of division, as K', will fall between A and K. Through K' pass a plane parallel to the base of P. The cross section thus made being equal and parallel to C, a parallelopiped, B', is formed.

As the altitudes of P and B' are commensurable,

$$P : B' :: A : K', \qquad \text{(B)}$$

Comparing the two proportions, (A) and (B),

$$B : B' :: K : K'. \quad \text{(Book v., Prop. xii.)}$$

As B is greater than B', the whole being greater than any of its parts, K must be greater than K'. On the contrary, by construction, it is less.

The supposition that leads to a false conclusion must be abandoned. It cannot be supposed that the fourth term of the desired proportion is greater than H. In similar manner, it may be proved that it is not less than H. The fourth term must, then, be H; that is,

$$P : B :: A : H.$$

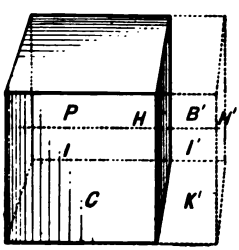
Conclusion: P and B being any parallelopipeds, etc.

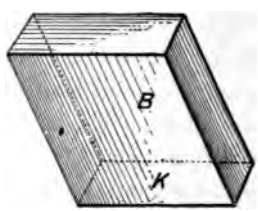
Corollary I. Triangular prisms with equivalent bases are proportional to their altitudes.

Corollary II. Prisms with equivalent bases are proportional to their altitudes.

PROPOSITION XI

491. Theorem: ***Parallelopipeds with equal altitudes are proportional to their bases.***





Statement: Let *P* and *B* be any two parallelopipeds with equal altitudes. They are then proportional to their bases.

Construction: Adjoining *P*, and between its edges produced, construct the parallelopiped *B'*, having its base *K'* equivalent to *K*. Draw *H H'* perpendicular to the planes it intersects.

Demonstration: *B'*, having the same altitude as *P*, has also the same altitude as *B*; and having, by construction, its base, *K'*, equivalent to *K*, it must be equivalent to *B*. (PROP. VIII.)

P and *B'* may be considered to be parallelopipeds with a common base *B*, and altitudes, *H* and *H'*; and, therefore,

$$P : B' :: H : H' \qquad \text{(PROP. X.)};$$

but, $C : K' :: L : L'$. (BOOK VI., PROP. XI.)

As $H H'$ and $L L'$ are straight lines included between parallel planes, $H : H' :: L : L'$ (BOOK VIII., PROP. XI.);

and, therefore, $P : B' :: C : K'$. (BOOK V., PROP. XI.)

By construction, however, B' is equivalent to B, and K' to K. By substitution, then, the last proportion becomes,

$$P : B :: C : K \qquad \text{(AXIOM I.)};$$

that is, the parallelopipeds are proportional to their bases.

Conclusion: P and B being any parallelopipeds, etc.

Corollary I. Triangular prisms with equal altitudes are proportional to their bases.

Corollary II. Any prisms with equal altitudes are proportional to their bases.

Exercises. 1. Find the length of the diagonal of a right rectangular parallelopiped whose edges are 5, 10, and 15.

2. What is the ratio of the volumes of two parallelopipeds with an altitude of 10, and bases, respectively, 5×6 and 5×7?

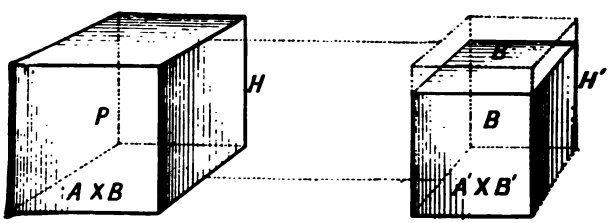
3. What is the ratio of the volumes of two parallelopipeds whose bases are 5×5, and whose altitudes are, respectively, 7 and 12?

4 Find the lateral surface of a right rectangular parallelopiped whose edges are 10, 20, and 30, the latter being the altitude.

5. Find the entire surface of a right rectangular parallelopiped whose edges are 8, 10, and 12.

PROPOSITION XII.

492. Theorem: ***Parallelopipeds are proportional to the products of their three dimensions.***



Statement: Let P and B be any two parallelopipeds, B, A, and H and B', A', and H' being their dimensions. Prove that,

$$P : B :: A \times B \times H : A' \times B' \times H'.$$

Construction: Produce the lateral edges of B so as to make a parallelopiped, B', of equal altitude with P.

Demonstration: P and B' have the same altitude (PROP. XI.); therefore, $P : B' :: A \times B : A' \times B'$. (BOOK VIII., PROP. XIV.)

Having the same base, $B' : B :: H : H'$ (PROP. X.), in which H is the altitude of B' or of P; then (BOOK V., PROP. XIV),

$$P : B :: A \times B \times H : A' \times B' \times H'. \qquad Q.\ E.\ D.$$

Conclusion: P and B being any two parallelopipeds, etc.

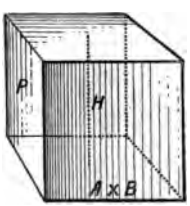
Corollary I. Triangular prisms are proportional to the products of their three dimensions.

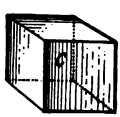
Corollary II. Prisms are proportional to the products of their bases and altitudes.

SECTION V.—VOLUMES.

PROPOSITION XIII.

493. Theorem: ***The volume of a parallelopiped is equal to the product of its three dimensions.***





Statement: Let P be any parallelopiped; $A \times B$, its base; and H, its altitude. Prove that $P = A \times B \times H$.

Construction.: Construct a cube, C, each of whose edges or dimensions is equal to the linear unit.

Demonstration: $P : C :: A \times B \times H : 1 \times 1 \times 1$ (PROP. XII., COR. II.); and, C being the solid unit (BOOK V., PROP. I.),

$$1 \times P = 1 \times A \times B \times H; \text{ or, } P = A \times B \times H.$$

Conclusion: P being any parallelopiped, it follows, etc.

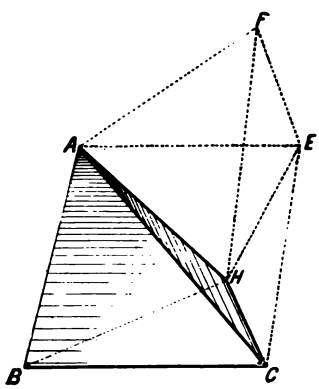
Corollary I. The volume of a prism is equal to the product of its base and altitude.

Corollary II. If prisms have equivalent bases, they are proportional to their altitudes ; if their altitudes are equal, they are proportional to their bases.

Corollary III. The volumes of similar prisms are proportional to the cubes of corresponding lines.

PROPOSITION XIV.

494. Theorem: ***The volume of a triangular pyramid is equal to one-third of the product of its base and altitude.***



Statement: Let *A–B C H* be any triangular pyramid. Its volume is equal to one-third of the product of its three dimensions.

Construction: Through *C* and *H*, draw *C E* and *H F*, equal and parallel to *A B*. Join *A* and *E*; also, *A* and *F*, and *E* and *F*. Draw *E H*.

Demonstration: The corresponding sides of the triangles *A E F* and *B C H* join the extremities of lines that are equal and parallel; they are themselves, therefore, equal and parallel. (BOOK III., PROP. III.) The figure *A E F–B C H* is, then, a triangular prism.

In the triangular prism thus constructed there are three pyramids: *A–B C H*, *H–A E F*, and *A–E C H*. These pyramids may be proved to be equivalent; thus, *A–B C H* and *H–A E F* are equivalent, as their altitudes are the same, being the altitude of the prism; and their bases are equal, being the bases of the prism. (PROP. IX.) The pyramid *H–A E F* may be called the pyramid *A–E H F*. This pyramid is equivalent to the pyramid *A–E C H*, as their altitudes are the same, being the perpendicular distance from *A* to *F C*; and their bases, *H E F* and *E H' C*, are equal, being (BOOK III., PROP. XVI.) halves of the parallelogram *F C*. (PROP. IX.) The three pyramids have thus been proved to be equivalent to one another; and any one of them, one-third of the prism.

The prism is equal to the product of its base and altitude. (PROP. XIII., COR. I.) The pyramid *A–B C H* is, then, equal to one-third of the product of its base and altitude.

Conclusion: *A–B C H* being any triangular pyramid, etc.

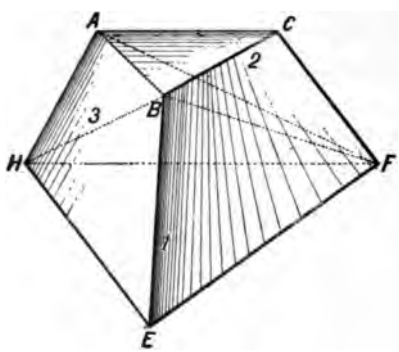
Corollary I. The volume of any pyramid is equal to one-third of the product of its base and altitude.

Corollary II. The volumes of similar pyramids are proportional to the cubes of corresponding lines.

Exercise. The distance between the centers of two circles is 12, the radii being 4 and 5. Find the common tangent.

PROPOSITION XV.

495. Theorem: ***The volume of any frustum of a triangular pyramid is equal to the volume of three pyramids having for their altitude the altitude of the frustum, and for their bases, the bases of the frustum and a mean proportional between the two bases of the frustum.***



Statement: Let *A B C–E H F* be any frustum of any triangular pyramid. The volume of the frustum equals the volume of three pyramids whose altitude is the altitude of the frustum, and whose bases are the bases of the frustum and a mean proportional between them.

Construction: Draw *B H*, *B F*, and *A F*, thus dividing the frustum into three pyramids, *B–H E F*, *F–A B C*, and *B–A H F*.

Demonstration: The pyramid *1* has for its base the lower base of the frustum, and for its altitude, the altitude of the frustum. The pyramid *2* has for its base the upper base of the frustum, and for its altitude, the altitude of the frustum.

Comparing pyramid *1* with pyramid *3*, they may be considered as having the same apex, *F*, with their bases in the same plane, *A E*. Having, then, the same altitude,

$$1 : 3 :: HEB : HAB$$ (PROP. XIV., COR. II.);

but *H E B* and *H A B* are triangles with equal altitudes, the distance between *A B* and *H E*. Then (BOOK VI., PROP. VI.),

$$HEB : HAB :: HE : AB;$$ and (BOOK V., PROP. XI.),

$$1 : 3 :: HE : AB.$$

In similar manner, it may be proved that

$$3 : 2 :: HF : AC.$$

A B C, being a cross section of the pyramid parallel to the base, is similar to the base (PROP. II., COR. I.);
and, therefore, $HE : AB :: HF : AC.$

$$1 : 3 :: 3 : 2$$ (BOOK V., PROP. XI.);

and, $$3 = \sqrt{1 \times 2}.$$ (BOOK V., PROP. I.)

Supposing the base of *1* to be *b* and of *2* to be *b'*, the altitudes being *a*, by substitution, this becomes,

$$3 = \sqrt{\tfrac{1}{3} a b \times \tfrac{1}{3} a b'}$$ (PROP. XIV.);

or, $$3 = \sqrt{\tfrac{1}{9} a^2 b b'}$$ or $$\tfrac{1}{3} a \sqrt{b b'}.$$

The third pyramid is, then, equal to one-third of the altitude of the frustum multiplied by a mean proportional between the bases of the frustum.

Conclusion: *A B C–H E F* being any frustum, etc.

Corollary I. The frustum of any pyramid is equal to the volume of three pyramids having for their altitude, etc.

MISCELLANEOUS EXERCISES.

496. 1. The volume of a triangular prism is equal to the product of one of its lateral faces by half the distance of this face from the opposite edge.

2. Find the edge of a cubical vessel that will hold one ton of water.

1 cu. ft. of water weighs 1,000 oz.

3. If mercury is fourteen times as heavy as water, what is the edge of a cube that will hold two tons of mercury?

4. How many square feet of tin will be required to cover the entire inside of a box, 4 ft. 3 in., by 5 ft. 2 in., by 6 ft. 6 in.?

5. What would be the cost of plastering a regular hexagonal cistern 12 ft. deep, each side of the base being 3 ft. long, at 10 cents per sq. ft.?

6. A pyramid 20 ft. high has 100 sq. ft. in its base; where must a cross section parallel to the base be placed to contain 55 sq. ft.?

7. If a pyramid is 15 ft. high, and its base contains 150 sq. in., what will be the area of a cross section parallel to the base at a distance of 5 ft. 5 in. from the base?

8. Find the volume of a regular pentagonal prism whose altitude is 12 ft., each side of the base being 10 ft. Of a similar solid, each side of base being 20 ft.

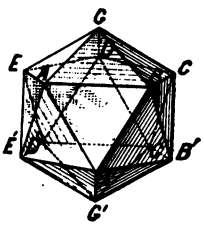
9. If, from a point 6 in. from the center of a circle whose radius is 3 in., two tangents are drawn, what is the length of the chord joining the points of contact?

10. What are the parts of a line 15 in. long, divided into extreme and mean ratios?

CONSTRUCTIONS.

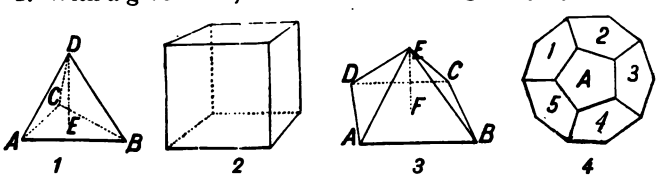
497. 1. Five, and only five regular convex polyhedrons can be constructed; viz., tetrahedron, hexahedron, octahedron, dodecahedron, and icosahedron.

1. Every convex polyhedral angle has at least three faces, and the sum of the plane angles is less than four right angles. These conditions are fulfilled by the following, and by no other figures :
2. Three, four, or five equilateral triangles, making the tetrahedron, octahedron, and icosahedron ;
3. Three squares, making the hexahedron ;
4. Three pentagons, making the dodecahedron.

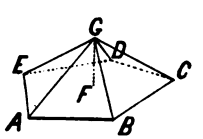


Icosahedron.

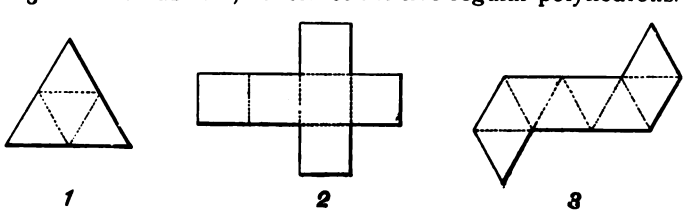
2. With a given line, construct the five regular polyhedrons.

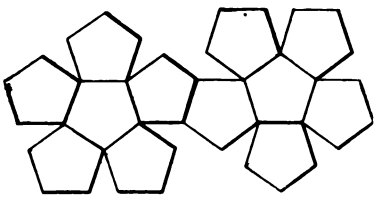


Construct, first, the equilateral base ; in *1, 3, 5,* erect a perpendicular from the center, and make the inclined lines equal to the given line.



3. With cardboard, construct the five regular polyhedrons.





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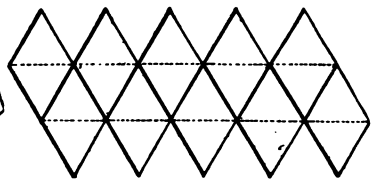


5

After folding, paste light paper over edges joined.

BOOK X.—SOLIDS WITH CURVED SURFACES.

SYNOPSIS.

SECTION I.—SECTIONS.

498. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. I. The right cross section of a cylinder is a circle ; the longitudinal section is a parallelogram.

PROP. II. The right cross section of a cone is a circle ; the longitudinal section is a triangle.

PROP. III. A section passing through the center of a sphere is a great circle ; any other section of a sphere is a small circle.

3. MISCELLANEOUS EXERCISES.

SECTION II.—SURFACES.

499. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. IV. The lateral surface of a cylinder is equal to the product of the circumference of a right section and the axis, or any element.

PROP. V. The lateral surface of a right cone is equal to half the product of the circumference of the base and the slant height.

PROP. VI. The surface of a sphere is equal to four great circles.

3. MISCELLANEOUS EXERCISES.

SECTION III.—VOLUMES.

500. 1. DEFINITIONS.

2. PROPOSITIONS

PROP. VII. The volume of a cylinder is equal to the product of its base and altitude.

PROP. VIII. The volume of a right cone is equal to one-third of the product of its base and altitude.

PROP. IX. The volume of a sphere is equal to the surface of the sphere multiplied by one-third of the radius.

SECTION IV.—POLARITY AND SYMMETRY.

501. 1. DEFINITIONS.

2. PROPOSITIONS.

PROP. X. If the vertices of one spherical triangle are poles of the opposite sides of another triangle, the vertices of the second triangle are poles of the opposite sides of the first.

PROP. XI. The sum of the angles of a spherical triangle is greater than two, and less than six right angles.

PROP. XII. Symmetrical spherical triangles are equivalent.

3. MISCELLANEOUS EXERCISES.

SECTION V.—SPHERICAL TRIANGLES.

502. 1. DEFINITIONS.

2. PROPOSITIONS.

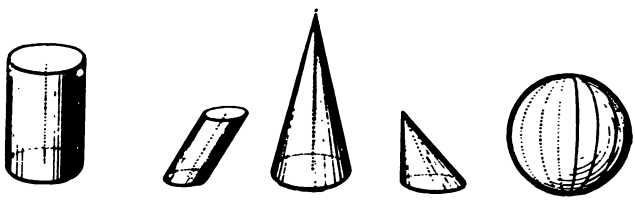
PROP. XIII. Spherical triangles on the same or on equal spheres are equal, when they have their parts, taken in the same order, equal, each to each.

PROP. XIV. Many propositions are equally true with spherical and plane triangles.

PROP. XV. The area of a spherical triangle is equal to its spherical excess multiplied by a tri-rectangular triangle.

3. MISCELLANEOUS EXERCISES.

SECTION I.—SECTIONS.



DEFINITIONS.

503. The **cylinder**, the **cone**, and the **sphere** are the *three round bodies*, or solids with curved surfaces.

504. A **spherical surface** is a surface that may be generated by the motion of the semi-circumference of a circle around its diameter as an axis.

505. A **diameter** of a sphere is any straight line passing through the center and terminating in both directions in the surface of the sphere.

506. A **sphere** is a solid bounded by a surface, all points of which are equally distant from a point which is the *center*.

507. A **cylindrical surface** is a surface that may be generated by the motion of any straight line around a second straight line to which it continues parallel.

The line that, by its motion, may generate the surface is, in any given position, an **element** of the surface.

508. A **cylinder** is a solid bounded by a cylindrical surface, and two parallel planes cutting all the elements. The two plane surfaces are called the *bases*.

509. A **circular cylinder** is a cylinder whose bases are circles.

510. The **axis of a cylinder** is the straight line that joins the centers of the bases and is parallel to all the elements.

511. A **right cylinder** is a cylinder whose axis is perpendicular to the bases.

512. An **oblique cylinder** is a cylinder whose axis is oblique to the bases.

513. A **conical surface** is a surface that may be generated by the motion of any straight line around another straight line which it meets or intersects.

The line that, by its motion, may generate the surface of the cone is, in any given position, an **element.**

514. A **cone** is a solid bounded by a conical surface, and a plane cutting all the elements.

515. The point in which all the elements meet is the **vertex**, or **apex.**

516. A **circular cone** is a cone whose base is a circle.

517. The **axis** of the cone is the straight line that joins the vertex with the center of the base.

518. A **right cone** is a cone whose axis is perpendicular to the base of the cone.

519. An **oblique cone** is a cone whose axis is oblique to the base of the cone.

520. The **cone of revolution** is a *right cone with circular base;* it may be generated by the revolution of a right-angled triangle about one of its sides as an axis.

521. The **cylinder of revolution** is a *right cylinder with circular base;* it may be generated by the revolution of a rectangle about one of its sides as an axis.

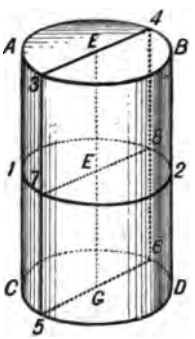
522. A **right cross section** of a cylinder or of a cone is a section made by a plane perpendicular to the axis.

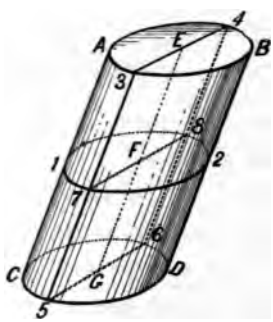
523. A **longitudinal section** of a cylinder or of a cone is the section made by a plane passing through any element.

Only one element can be drawn through a given point in the surface of a cylinder (POST. III.), or of a cone. (POST. I.)

PROPOSITION I.

524. Theorem: ***Any longitudinal section of any cylinder is a parallelogram. Any section of a circular cylinder, parallel to the base, is a circle.***





I. Statement: Let *A D* be any cylinder; *3 5*, any element; and *3 6*, any section passed through *3 5*.

Construction: In *3 6*, draw a line from *4* parallel to *3 5*.

Demonstration: The line drawn from *4* is an element (POST. III.); and, being in the plane *3 6*, it is *4 6*, the intersection of *3 6* with the cylindrical surface. *4 6* and *3 5* being, then, parallel, they are equal (BOOK VIII., PROP. X); and *3 6* is, therefore, a parallelogram. (BOOK III., PROP. III.)

II. Statement: Let *A D* be any circular cylinder; and *1 2*, any section parallel to *A B* and *C D*. Prove *1 2* a circle.

Construction: Draw the axis *E G*. Through the axis pass the plane *3 6*, intersecting the surface of *A D* in *3 5* and *4 6*.

Demonstration: The longitudinal sections *3 6*, *3 8*, and *7 6* have been proved to be parallelograms. *E* and *G* being centers of circles, the line *E G*, as a diameter, divides each of these parallelograms into two equal parallelograms. (Book III., Prop. XVIII.) The lines *F 7* and *F 8* are, then, equal to the equal radii, *E 3* and *E 4*, or *G 5* and *G 6*; and these lines are all equal. (Axiom I.) As *7 8* may be any line in *1 2*, and it is bisected at *F*, *1 2* is a circle, by definition.

Conclusion: *A D* being any cylinder fulfilling the given conditions, it follows, in general, etc.

Corollary I. The longitudinal section of a right cylinder is a rectangle.

Corollary II. The bases of any cylinder are equal. (Superposition.)

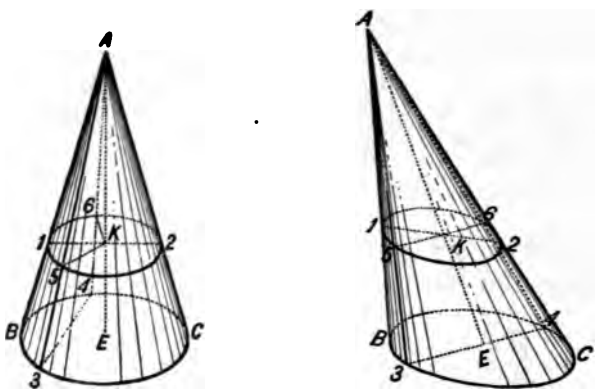
Corollary III. The bases of any circular cylinder are equal circles.

Corollary IV. A section parallel to the base of a cylinder is equal to the base.

Corollary V. Parallel sections of any cylinder cutting all the elements are equal.

PROPOSITION II.

525. Theorem: ***The longitudinal section of any cone is a triangle; any section of a circular cone, parallel to the base, is a circle.***



I. Statement: Let *A–B C* be any cone; *A 3*, any element; and *A 3 4*, the section of a plane passed through *A 3*.

Construction: Join *A 4* by a straight line.

Demonstration: *A 3* is a straight line, by hypothesis; and *3 4*, because it is the intersection of two planes. (BOOK VIII., PROP. VI.) *A 4*, being a straight line, is an element (POST. I.); it, therefore, lies in the surface, and it is the intersection of the plane *A 3 4* with the conical surface. The figure *A 3 4*, having three straight sides, is a plane triangle.

II. Statement: Let *A B C* be any circular cone; and *1 2*, a section parallel to the base.

Construction: Draw the axis *A K E*, and through it pass the planes, *A E 3* and *A E 4*, intersecting *1 2* in any lines, *5 K* and *6 K*.

Demonstration: *A K 5*, *A K 6*, *A E 3*, and *A E 4* are triangles. *K 5* is parallel to *E 3*; and *K 6*, to *E 4*. (Book VIII., Prop. X., Cor. II.) Then (Book VII., Prop. VI.),

$$5K : 3E :: AK : AE,$$

and,

$$6K : 4E :: AK : AE;$$

therefore,

$$5K : 3E :: 6K : 4E.$$ (Book V., Prop. XI.)

As *6 K* equals *4 E*, *5 K* equals *3 E*. Any two lines drawn to the circumference from *K* being equal, *1 2* is a circle.

Conclusion: As *A-B C* is any cone fulfilling the given conditions, it follows, in general, that, etc.

Corollary I. The axis of a circular cone passes through the centers of all sections parallel to the base.

Corollary II. Sections parallel to the base are proportional to the square of the distance from the apex.

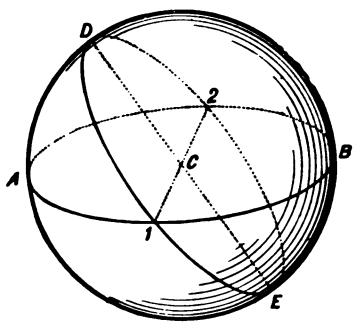
Corollary III. Right cross sections are proportional to the square of the distance from the apex.

Corollary IV. In cones with equal bases and altitudes, sections parallel to the bases and at equal distances from the apices, or from the bases, are equal.

Corollary V. In cones with equal bases and altitudes, right cross sections at equal distances from the apices, or from the bases, are equal.

PROPOSITION III.

526. Theorem: ***A section passing through the center of a sphere is a great circle; any other section of a sphere is a small circle.***



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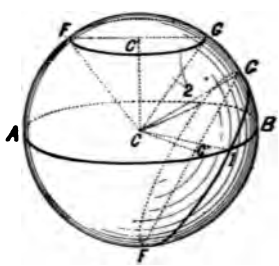


2

Statement: Let the sphere whose center is *C* be any sphere, and let *A B* be any section passing through the center; and *F G*, any section of the sphere not passing through its center. *D E* is to be proved a great circle, or a circle having for its radius the radius of the sphere; and *F G*, a small circle, or a circle with a radius less than the radius of the sphere.

Construction: Draw *C D*, *C E*, *C 1*, and *C 2* in figure *1*. Draw *C C'* perpendicular to the plane of the section *F G*, in figure *2*; and then draw *C' F*, *C' G*, etc.

Demonstration: I. In figure *1*, all the points, as *D*, *1*, *E*, *2*, etc., in the perimeter of the section are points in the surface of the sphere. These points, by the definition of a sphere, are all equally distant from the center *C*. As, however, *C* and the points *D*, *1*, *E*, *2*, etc., are all, by construction, in the same

plane, the section must be a circle. It has for its radius the radius of the sphere, and it is, therefore, a great circle.

II. In figure *2*, the triangles *C C' F*, *C C' G*, etc., are all right-angled at *C'*, by construction. The hypotenuses *C F*, *C G*, etc., are all equal, being radii of the sphere. The lines *C' F*, *C' G*, etc., are, then, all equal. (BOOK II., PROP. III.) As all the points in the perimeter of the section have, thus, been proved equally distant from the point *C'* in the plane of the section, the section must be a circle. *C' G*, the radius of the section, is less than *C G*, the radius of the sphere. (BOOK IV., PROP. VI.) The section of the sphere not passing through the center having a radius less than the radius of the sphere, the section is called a small circle.

Conclusion: As the sphere whose center is, etc.

Corollary I. All sections of spheres are circles.

Corollary II. All great circles are equal.

Corollary III. All great circles bisect the sphere and its surface.

Corollary IV. All great circles bisect each other.

Corollary V. The poles of a section of a sphere are equally distant from every point of its circumference.

Corollary VI. The polar distances of all the points in the circumference of a section of a sphere from either pole are equal.

Corollary VII. Sections at equal distances from the centers of the same or of equal spheres are equal.

Corollary VIII. Sections are proportional to the products of the segments of the diameter drawn perpendicular to them.

Corollary IX. A plane perpendicular to a radius at its outer extremity is tangent to the sphere.

Corollary X. A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.

MISCELLANEOUS EXERCISES.

527. 1. Prove that the sum of the squares of three chords at right angles to one another from any point in the surface of the sphere is equal to the square of the diameter.

2. The rectangles of secants and their respective external segments are equal, if the secants are from the same point.

3. The rectangle of a secant by its external segment is equal to the square of the tangent from the same point.

4. The rectangles of the segments of chords intersecting at the same point are equal.

5. Find the locus of the centers of spherical sections made by planes that contain a given straight line.

6. Find the locus of the centers of spherical sections which pass through a given point.

7. Construct a sphere of given radius, which shall pass through two given points and be tangent to a given plane.

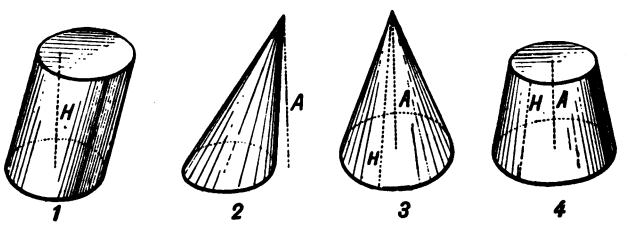
8. Construct a sphere of given radius, the surface passing through three given points.

9. Construct a sphere of given radius, and tangent to three given planes.

10. What is the area of a section, 5 inches from the center of a sphere, the radius of the sphere being 7 inches?

11. How much larger than the section in the last exercise is one 4 inches from the center.

SECTION II.—SURFACES.



DEFINITIONS.

528. The altitude of a cylinder is the perpendicular distance between its bases; as *H*, in figure *1*.

529. The altitude of a right cylinder is equal to the axis or to any one of the elements of the cylinder.

530. The altitude of a cone is the perpendicular distance between its apex and its base; as *A*, in figure *2*.

531. The altitude of a right cone is equal to the axis; as *A*, in figure *3*.

532. The **slant height** of a right cone is any one of its elements; as *H*, in figure *3*.

533. The **frustum of a cone** is a portion of a cone included between the base and any section parallel to the base; as the figure *4*.

534. The **altitude** of the **frustum** of a **cone** is the perpendicular distance between the bases; as *A*, in figure *4*.

535. The **slant height** of the **frustum** of a **right cone** is the part of any element between the bases; as *H*, in figure *4*.

536. A prism is inscribed in a cylinder, when the bases of the prism are inscribed in the bases of the cylinder and the lateral edges of the prism are elements in the lateral surface of the cylinder.

The diagram on the opposite page illustrates an inscribed prism.

537. A prism is circumscribed about a cylinder, when the bases of the prism are circumscribed about the bases of the cylinder and the lateral edges of the prism are elements in the lateral surface of the cylinder.

538. A pyramid is inscribed in a cone, when the apices coincide, the base of the pyramid is inscribed in the base of the cone, and the lateral edges of the pyramid are elements in the lateral surface of the cone.

539. A pyramid is circumscribed about a cone, when the apices coincide, the base of the pyramid is circumscribed about the base of the cone, and the lateral edges of the pyramid are elements in the surface of the cone.

540. A frustum of a pyramid is inscribed or circumscribed about a frustum of a cone, when the bases of the frustum of the pyramid are inscribed or circumscribed about the bases of the frustum of the cone and the lateral edges of the frustum of the pyramid are elements in the lateral surface of the frustum of the cone.

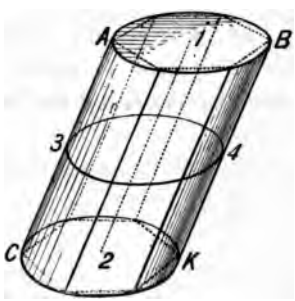
541. A **zone** is a portion of the surface of a sphere included between two parallel planes.

542. The **projection** of one line upon another is the part of the second line included between perpendiculars let fall upon it from the extremities of the first line.

What is the projection of a figure upon a plane?

PROPOSITION IV.

543. Theorem: ***The lateral surface of a cylinder is equal to the product of the circumference of a right section and the axis, or any element.***



Statement: Let *A B C K* be any circular cylinder. Let *3 4* be the circumference of any right cross section, and let *1 2* be the axis. The lateral surface is to be proved equal to the product of *3 4* and *1 2*.

Construction: In the cylinder, inscribe a regular prism.

Demonstration: The lateral surface of the prism is equal to the perimeter of a right section multiplied by any one of the lateral edges. (BOOK IX., PROP. IV.) If the number of faces of the regular prism be increased indefinitely, the bases of the prism become the bases of the cylinder; the cross section of the prism, the cross section of the cylinder; the lateral surface of the prism, the lateral surface of the cylinder; the edges become elements; and, in its entirety, the prism becomes the cylinder. The lateral surface of the cylinder is, then, equal to the circumference of a right section, as *3 4*, multiplied by any element, or by the axis, as the elements and the axis are all equal and parallel straight lines.

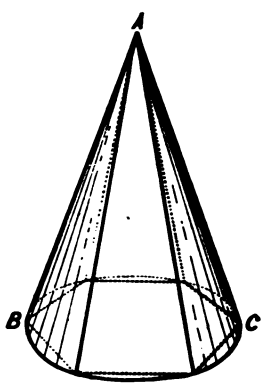
Conclusion: As *A K* is any cylinder, etc.

Corollary I. The lateral surface of a right cylinder is equal to the product of the circumference of the base by the axis, or by any element.

Corollary II. The entire surface of a cylinder is equal to the lateral surface and the areas of the two bases.

PROPOSITION V.

544. Theorem: ***The lateral surface of a right circular cone is equal to half of the product of the circumference of the base and the slant height.***



Statement: Let *A–B C* be any right cone, and let *B C* be the circumference of the base; and *A B*, the slant height. The lateral surface is equal to half of the product of the circumference of the base and the slant height.

Construction: In the cone, inscribe a regular pvramid.

Demonstration: The lateral surface of the regular inscribed pyramid is equal to half of the product of the circumference of its base and its slant height. (BOOK IX.,

PROP. V.) If the number of faces of the regular inscribed pyramid be increased to infinity, the base of the pyramid becomes the base of the cone, the lateral surface of the pyramid becomes the lateral surface of the cone, the slant height of the pyramid becomes the slant height of the cone, and, in its entirety, the pyramid becomes the cone. The theorem, then, applies to the cone as well as to the pyramid, and the lateral surface of the cone is equal to half of the product of the circumference of the base and the slant height.

Conclusion: *A–B C* being any right cone, it follows, in general, that: The lateral surface of a right cone is equal to half of the product of the circumference of the base and the slant height.

Corollary I. The entire surface of a right circular cone is equal to the lateral surface and the area of the base.

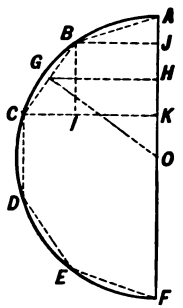
Corollary II. The lateral surface of the frustum of a right circular cone is equal to half of the sum of the circumferences of the two bases multiplied by the slant height.

Corollary III. The entire surface of the frustum of a cone is equal to the lateral surface and the sum of the two bases.

Corollary IV. If one line be revolved about another line in the same plane, without changing the distance or inclination to each other in the plane, the surface generated by it is equal to the product of the revolving line and the circumference described by its middle point. The surface generated is the surface of the frustum of a cone, and the circumference described is equal to half of the sum of the circumferences of the bases.

PROPOSITION VI.

545. Theorem: ***The surface of a sphere is equal to four great circles.***



Statement: The surface of a sphere is equal to four circles, each having for its radius the radius of the sphere.

Construction: Construct a semi circumference ***A C F***, and in it inscribe a regular semi-polygon, ***A B C D E F***. Draw ***O G*** perpendicular to ***B C***; ***B J*** and ***C K*** perpendicular to ***A F***; and ***B I*** perpendicular to ***C K***.

Demonstration: If the figure ***A C F*** is revolved about ***A F*** as an axis, the semicircle will generate a sphere; and the regular semi-polygon, a regular inscribed polyhedron.

The surface generated by any part of the perimeter of the regular semi-polygon, as ***B C***, is equal to the product of ***B C*** and the circumference described by its middle point, ***G***; that is, the circumference ***H G***. (PROP. V., COR. IV.)

Since the triangles ***C B I*** and ***G O H*** have their sides respectively perpendicular to one another, they are similar

(BOOK VII., PROP. V.) ; and, therefore,

$$CB : (BI \text{ or }) JK :: GO : GH :: \text{Circ. } GO : \text{Circ. } GH;$$

then, $CB \times$ Circumference $GH = JK \times$ Circumference GO. (BOOK V., PROP. I.)

The surface generated, then, by any part of the regular semi-perimeter is equal to the product of the projection of that part on the axis and the circumference of the inscribed circle.

The surface generated by the entire semi-perimeter is, by the addition of its parts, equal to the product of the entire axis by the circumference of the inscribed circle. This proposition is true, whatever be the number of sides in the regular semi-perimeter.

If the number of sides of the regular semi-perimeter is increased to infinity, the semi-perimeter becomes the semi-circumference, the inscribed circle becomes a great circle, and the solid generated is a sphere, the axis being one of its diameters. The theorem, then, applies to the sphere ; and its surface is equal to the product of the axis, or diameter, and the circumference of a great circle.

The diameter equals $2R$, and the circumference of the great circle equals $2\pi R$. The product is, then, $4\pi R^2$. As πR^2 is the area of a great circle, the surface of the sphere equals four great circles.

Conclusion : The semi-circumference, $ABCDEF$, etc.

Corollary I. The area of a zone is equal to the product of its projection upon its axis and the circumference of a great circle.

Corollary II. The surface of a sphere is equal to the lateral surface of the circumscribed cylinder.

Corollary III. The surface of a sphere is to the surface of the circumscribing cylinder as *2* is to *3*.

MISCELLANEOUS EXERCISES.

546. 1. If two straight lines are tangent to a sphere at the same point, the plane determined by them is tangent to the sphere.

2. Prove that the surface of a sphere is equal to the lateral surface of its circumscribed cylinder.

3. The locus of points, the sum of the squares of the distances of which from two fixed points are constant, is a sphere.

4. Construct a sphere of given radius which shall pass through a given point and be tangent to two given spheres.

5. Construct a cylinder, the altitude being given, and the convex surface being equivalent in area to the two bases.

6. Find the locus of all the points which are at a given distance from the surface of a given cylinder of revolution.

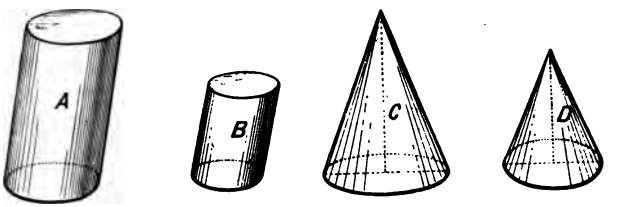
7. Find the locus of all the points, the distances of each one of which from two given spheres of equal radii are equal.

8. Find the locus of all the points at a given distance from a given straight line.

9. Find the lateral surface, and the entire surface, of a cylinder of revolution whose altitude is 10, the radius of the base being 6.

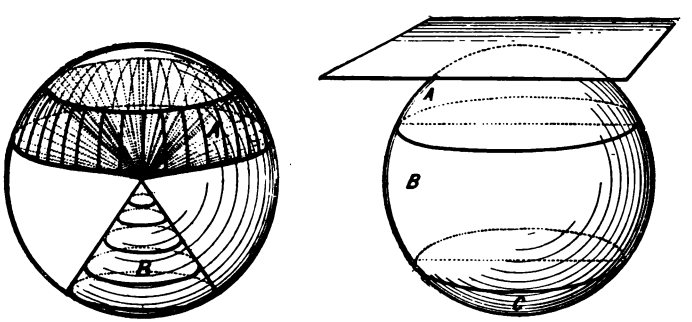
10. Find the lateral surface, and the entire surface, of a frustum of a cone of revolution, the radii of the bases being 5 and 10 and the altitude being 4.

SECTION III.—VOLUMES.



DEFINITIONS.

547. Similar cylinders and **cones** are those whose axes are equally inclined to the bases and whose axes are proportionate to the diameters of the base; thus, *A* and *B* are similar cylinders, and *C* and *D* are similar cones.



548. A **spherical segment** is a portion of a sphere included between parallel planes; thus, in the right-hand figure above, *A*, *B*, and *C* are spherical segments.

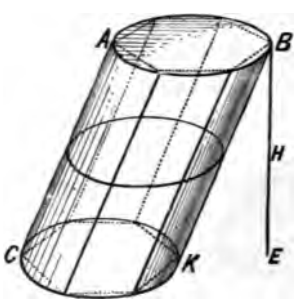
549. A **spherical sector** is a volume that may be generated by the revolution of the sector of a circle about a straight line in its plane as an axis; thus, in the left-hand figure, *A* and *B* are spherical sectors.

550. A polyhedron is said to be inscribed in a sphere when the vertices of all of its solid angles are in the surface of the sphere.

551. A polyhedron is said to be circumscribed about a sphere when all of its faces are tangent to the sphere.

PROPOSITION VII.

552. Theorem: ***The volume of a cylinder is equal to the product of its base and altitude.***



Statement: Let *A B C K* be any cylinder; *B K*, its base; and *H*, its altitude. The volume of *A B C K* is equal to the product of *B K* and *H*.

Construction: Inscribe a regular prism in the cylinder.

Demonstration: The volume of the prism is equal to the product of its base and altitude. (BOOK IX., PROP. XIII.) This theorem is true, whatever be the number of sides of the prism. If the number of sides of the regular inscribed prism be increased to infinity, the bases of the prism become the

bases of the cylinder, the lateral surface of the prism the lateral surface of the cylinder, and, in its entirety, the prism becomes the cylinder, the altitudes remaining the same. The theorem, then, applies to the cylinder as well as to the prism, and the volume of the cylinder is equal to the product of its base and altitude.

Conclusion: *A B C K* being any cylinder, etc.

Corollary I. The volume of a right cylinder is equal to the product of its base and axis, or, of its base and any element.

Corollary II. Cylinders are to each other as the products of their bases and altitudes.

Corollary III. If cylinders have equal bases, they are proportionate to their altitudes; if they have equal altitudes, they are proportionate to their bases.

Corollary IV. Similar cylinders are proportionate to the cubes of any homologous lines.

Exercises. 1. Find the entire surface of a right circular cylinder whose altitude is 10 inches, the radius of the base being 5 inches.

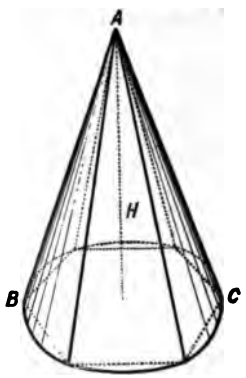
2. Find the entire surface of a right circular cone whose altitude is 10 inches, the radius of the base being 5 inches.

3. What is the entire surface of the frustum of a right circular cone whose altitude is 10 inches, the radii of the bases being 5 inches and 6 inches?

4. What is the volume of a cylinder whose altitude is 12 feet, the radius of the base being 3 feet 4 inches?

PROPOSITION VIII.

553. Theorem: ***The volume of a right circular cone is equal to one-third of the product of its base and altitude.***



Statement: Let *A–B C* be any right cone; *B C*, its base; and *H*, its altitude. The volume of *A–B C* is equal to one-third of the product of *B C* and *H*.

Construction: Inscribe a regular pyramid in the cone.

Demonstration: The volume of the pyramid is equal to one-third of the product of its base and altitude, whatever be the number of its faces. If the number of the faces of the inscribed pyramid be increased to infinity, the pyramid, in its entirety, becomes the cone. The theorem, then, applies to the cone as well as to the pyramid, and the volume of the right cone is equal to one-third of the product of the base and altitude.

Conclusion: *A–B C* being any right circular cone, etc.

Corollary I. Right cones are to each other as the products of their bases and altitudes.

Corollary II. If cones have equal bases, they are proportionate to their altitudes; if they have equal altitudes, they are proportionate to their bases.

Corollary III. Similar cones are proportionate to the cubes of any homologous lines.

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554. Exercises. 1. Find the volume of a right circular cone, the radius being 5 ft. 5 in., and the slant height, 6 ft.

2. What is the volume of a circular cylinder 14 ft. long, the radius of the base being 2 ft.?

3. How high is a circular cylinder that holds 50 gallons, the radius of the base being 1 ft. 8 in.?

A gallon contains 231 cu. in.

4. How many bushels of coal will the frustum of a right square pyramid hold, the edges of the bases being 30 ft. and 20 ft., and the slant height, 12 ft.?

A bushel contains 2,150.42 cu. in.

5. The frustum of a right circular cone, the diameters of the bases being 10 ft. and 12 ft., and the altitude, 4 ft., will hold how many pounds of wheat?

A bushel equals 60 lb.

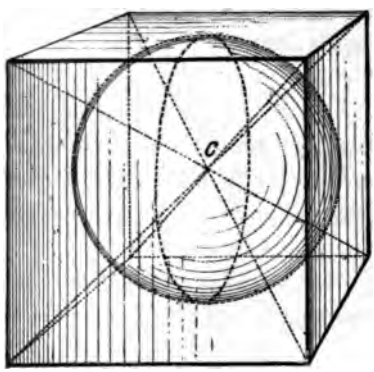
6. How many tons of coal will a pocket of the form and dimensions given in the preceding exercise hold?

A ton equals 40 cu. ft.

7. How many cubic yards are there in a tunnel one mile long, the cross section being a semicircle with radius, 8 ft.?

PROPOSITION IX.

555. Theorem: ***The volume of a sphere is equal to the surface of the sphere multiplied by one-third of the radius.***



Statement: The volume of a sphere is equal to its surface, four great circles, multiplied by one-third of its radius.

Construction: About the sphere circumscribe a cube, and draw straight lines from the center of the sphere to the polyhedral angles of the cube.

Demonstration: The volume of any pyramid is equal to one-third of the product of its base and altitude. By construction, the bases of the pyramids in the figure are all tangent planes, and the altitudes of the pyramids thus constructed are, therefore, the same; viz., the radius of the sphere. (PROP. III., COR.) The sum of the volumes of the pyramids is, then, equal to the surface of the circumscribed solid multiplied by one-third of the radius. (BOOK IX., PROP.

XIV.) This will be true, whatever be the number of tangent planes in the surface of the circumscribing polyhedron. The number of faces or tangent planes of the circumscribing polyhedron may be increased by passing tangent planes through the intersections of the surface of the sphere with the radii drawn to the vertices of the polyhedral angles.

If this process is continued without limit, the circumscribing polyhedron will become the sphere and the surface of the polyhedron will become the surface of the sphere. The volume of the sphere is, then, equal to its surface, or the aggregate of the bases of the pyramids multiplied by one-third of the radius of the sphere, or the common altitude of the pyramids. (BOOK IX., PROP. V.)

Conclusion: As the sphere whose center is *C*, etc.

Corollary I. The volume of a sphere equals $\frac{4}{3}\pi R^3$, or $\frac{1}{6}\pi D^3$.

Corollary II. The volumes of spheres are proportionate to the cubes of their radii, or to the cubes of their diameters.

Corollary III. The volume of the sphere is to the volume of the circumscribing cylinder as *2* is to *3*.

Corollary IV. The volume of a spherical sector is equal to the zone that forms its base, multiplied by one-third of the radius of the sphere.

Corollary V. The volume of a spherical segment is equal to the corresponding spherical sector and the sum or difference of the corresponding cones, according as the segment does or does not include the center of the sphere.

556. Exercises. 1. Find the surface of a sphere whose diameter is 4 feet.

2. Find the volume of a sphere the diameter being 4.

3. What is the area of a zone of the earth's surface, the radius being 4,000 miles and the projection on the axis being 1,000 miles?

4. If two boxes are similar, and one, with dimensions 2 ft., 4 ft., and 5 ft., holds one-fourth as much as the second, what are the dimensions of the second?

5. The volumes of two similar cones are 27 cu. ft. and 1,000 cu. ft. The height of the first is 12 ft.; what is the height of the second?

6. The circumference of a bowl in the form of a hemisphere is 30 in. What is the area of the surface?

7. At what distance may mountains 5 miles high be seen, if the earth is an exact sphere whose radius is 4,000 miles, and making no allowance for refraction of the atmosphere?

8. What is the ratio of the surface to the volume of a sphere whose diameter is 10 ft.?

9. Find the number of square inches of tin in a funnel, the altitude being 5 in., and the diameters of the bases 10 in. and 6 in.

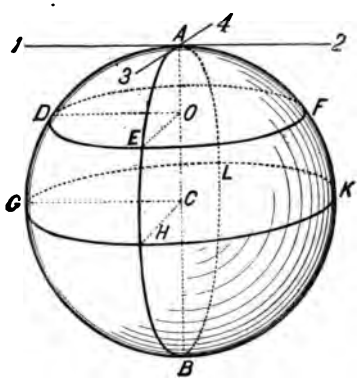
10. The edge of a cube is 10 in., what is the volume of the inscribed sphere?

11. What is the surface of the inscribed sphere in the last exercise?

12. If a sphere has a diameter of 20 in., what is the surface of a polar zone, the projection on its axis being 5 in.?

13. If each edge of a cube is 24 in., what is the volume of a polar segment of the inscribed sphere, the projection on the axis being 8 in.?

SECTION IV.—POLARITY AND SYMMETRY.



DEFINITIONS.

557. The diameter of a sphere, perpendicular to the plane of any circle, great or small, is the **axis** of that circle, or of any arc of the circle; thus, ***A B*** is the axis of the small circle ***D F;*** and, also, of the great circle ***G K***.

558. The extremities of the axis of any arc or circle are the **poles** of the arc or circle; thus, ***A*** and ***B*** are the poles of the circles ***D F*** and ***G K;*** and, of all their arcs.

Parallel circles or arcs have the same axes and poles.

559. A **spherical angle** is an angle on the surface of a sphere formed by the intersection of the arcs of two great circles; thus, ***D A E*** is a spherical angle, and ***A*** is its vertex.

Is ***D E A*** a spherical angle? ***G H E?*** ***D E H?*** ***G H B?*** Point to other spherical angles. Construct several spherical angles.

560. The measure of a spherical angle is the included arc of its great polar circle; thus, *G H* is the measure of *A*, or of *B*.

This is proved by methods entirely similar to those employed with other commensurable and incommensurable quantities.

The spherical angle and the dihedral angle formed by the planes of its two great circles are, then, evidently equal.

561. The plane angle formed by the intersection of the two great circles with any one of the polar circles may be taken as the measure of the dihedral angle, or of the spherical angle, as its sides are perpendicular to the axis, which is the edge of the dihedral angle. The plane angle formed by the tangents to the great circles at the vertex may also be taken as the measure of the dihedral angle, or of the spherical angle (BOOK VIII., DEF.); thus, any one of the angles *1 A 3*, *D O E*, *G C H*, may be taken as the measure of the dihedral angle formed by the planes of the two great circles, or of the spherical angle, *D A E*.

The included arcs of the small polar circles may, in different problems, be taken as measures of the equal plane angles, but the included arcs of great polar circles are the only arcs that serve as measures for spherical angles.

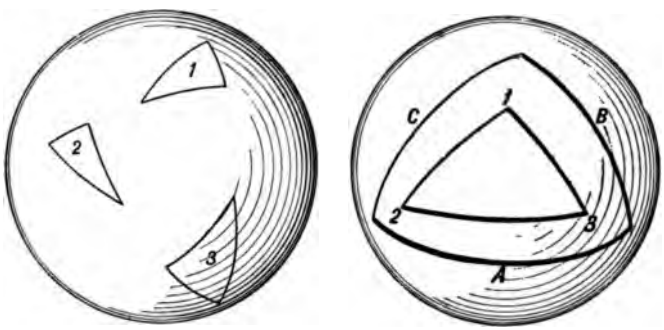
562. A **spherical polygon** is a portion of the surface of a sphere bounded by the intersecting arcs of great circles, each arc being less than a semi-circumference.

The arcs of small circles do not serve as sides of spherical polygons.

563. A **spherical triangle** is a spherical polygon with three sides.

Spherical triangles are classified in the same manner as plane triangles; thus, an *isosceles spherical triangle* is a spherical triangle with two of its sides equal.

564. Equal spherical triangles or polygons, on the same or equal spheres, are triangles or polygons whose sides and angles, taken in the same order, are equal, each to each; thus, *1* and *2* are equal spherical triangles.



565. Triangles and polygons having sides and angles, taken in the reverse order, equal, each to each, are said to be **symmetrical triangles** and **polygons.**

See *A B C* and *D E F*, page 394.

566. Polar triangles are triangles the vertices of one of which are poles of the opposite sides of the other; thus, the triangles in the right-hand figure are polar triangles, the vertex *1* being the pole of *A*; *2*, of *B*; *3*, of *C*.

Each pole is at the distance of a quadrant from the arc of which it is the pole.

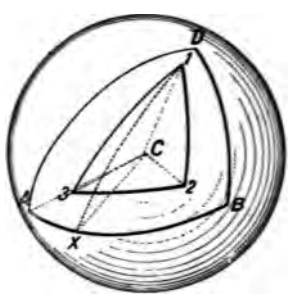
567. Polar distance between points, or points and lines, is the distance between them measured on the arc of a great circle.

568. Every arc of the circumference of a great circle is the measure of a plane angle at the center of the sphere; thus, *G H* is the measure of *G C H*, in the figure on page 387.

PROPOSITION X.

569. Theorem: *If the vertices of one spherical triangle are poles of the opposite sides of another triangle, the vertices of the second triangle are poles of the opposite sides of the first triangle.*

Statement: Let the vertices of the spherical triangle *1 2 3* be poles of the respectively opposite sides of the triangle *A B D*. The vertices of the triangle *A B D* are then the poles of the respectively opposite sides of the triangle *1 2 3*.



Construction: Let straight lines be drawn from *C*, the center of the sphere, to *1* and *2*, and to any point of *A B*, as *X*. Pass the arc of a great circle from *1* to *X*.

Demonstration: As *A B* is the side of a spherical triangle, it is the arc of a great circle, and its plane must pass through the center of the sphere. The radius *C X* must, then, be in the plane of *A B*. The point *1* being the pole of *A B*, the radius *1 C* is, by definition, perpendicular to the plane of *A B*. If *1 C* is perpendicular to the plane of *A B*, it is perpendicular to *C X*, which passes through its foot. The lines *1 C* and *C X* are, then, two radii at right angles, and the arc *1 X* is a quadrant. (BOOK VI., PROP. I., COR. III.) As *X* is any point, *1* is at a quadrant's distance from every point of *A B*.

In similar manner, *2* is a quadrant distant from every

point of *A D*, and *3* is a quadrant distant from every point of *D B*.

It has, thus, been shown that *1* and *2* are each at a quadrant's distance from *A*. The angles *A C 1* and *A C 2* must, then, be right angles, and *A C* is perpendicular to *C 1* and *C 2*. It is, then, also perpendicular to their plane; that is, the plane of *1 2*. The point *A* being, then, the extremity of the diameter perpendicular to the plane of *1 2*, it is, by definition, the pole of *1 2*. (BOOK VIII., PROP. IV.)

In similar manner, as it has been shown that *1* and *3* are each a quadrant distant from *B*, it may be proved that *B* is the pole of *1 3*; and, as it has been shown that *2* and *3* are each a quadrant distant from *D*, it may be proved that *D* is the pole of *A B*. *A*, *B*, and *D* are, then, the poles of the respectively opposite sides of the other triangle.

Conclusion: As *1 2 3* is any spherical triangle, etc.

Corollary I. The pole of a circle, great or small, is equally distant from all the points in the circumference of the circle.

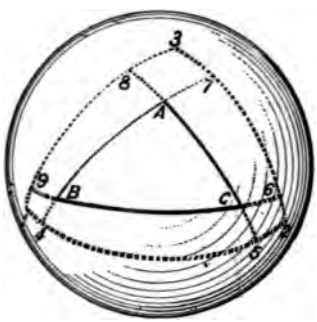
Corollary II. The poles of equal circles, on the same or on equal spheres, are equally distant from the circumferences of these circles.

Corollary III. The pole of the side of a spherical polygon is at a quadrant's distance from every point in the side.

Corollary IV. If a point is at a quadrant's distance from each of two points of the arc of a great circle, it is the pole of that circle.

PROPOSITION XI.

570. Theorem: ***The sum of the angles of a spherical triangle is greater than two, and less than six right angles.***



Statement: Let *A B C* be any spherical triangle. The sum of its angles is less than six, and greater than two right angles.

Construction: Construct the polar triangle *1 2 3*. Produce the sides of *A B C* to *4*, *5*, *6*, *7*, *8*, and *9*, points in the polar triangle.

Demonstration: Any one of the angles of a spherical triangle is measured by the supplement of the opposite side of the polar triangle; thus, *A* is measured by the supplement of *1 2*. This may be proved by showing that *4 5*, the included arc of the great polar circle of *A*, and, therefore, the measure of *A*, is the supplement of *1 2*. The arc *1 5* is a quadrant, as *1* is the pole of *A C*. (PROP. X., COR. II.) The arc *2 4* is a quadrant, as *2* is the pole of *A B*. The arcs *1 5* and *2 4*, then,

are together equal to a semi-circumference. If from their sum *1 2* be subtracted, *4 5* will remain ; that is, *4 5* is the supplement of *1 2*. The arc *4 5*, then, which is the measure of *A*, has been proved to be the supplement of *1 2*, the opposite side of the polar triangle.

Each angle of the spherical triangle is, then, measured by a semi-circumference, less the opposite side of the polar triangle. The three angles of the spherical triangle must, then, be measured by three semi-circumferences, less the sum of the sides of the polar triangle.

As the semi-circumference is the measure of two right angles, each angle of the spherical triangle is less than two right angles, and the three angles together are less than six right angles.

The sides of the polar triangle are measures of corresponding plane angles at the center of the sphere. As these three plane angles are less than four right angles (BOOK VIII., PROP. XV.), their measures, the sides of the polar triangle, are together less than the circumference.

As the arcs to be subtracted from three semi-circumferences are less than a circumference, the remainder must be greater than a semi-circumference. That is, the measure of the angles of the spherical triangle must be greater than a semi-circumference. The sum of the angles is, then, greater than two right angles ; and it has been proved to be less than six right angles.

Conclusion : As *A B C* is any spherical triangle, etc.

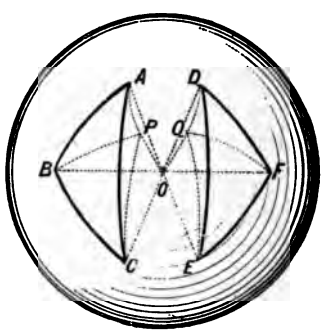
Corollary I. An angle of a spherical triangle is equal to the supplement of the opposite side of the polar triangle.

Corollary II. The sides of a spherical polygon are together less than the circumference of a great circle.

Corollary III. The sum of any two of the sides of a spherical triangle is greater than the third side.

PROPOSITION XII.

571. Theorem: ***Symmetrical spherical triangles are equivalent.***



Statement: Let *A B C* and *D E F* be any two symmetrical triangles, having *A B* = *D F*, *A C* = *D E*, and *B C* = *E F*; that is, the sides equal, each to each, taken in the reverse order. Prove the triangles equivalent.

Construction: Find *P*, the pole of the small circle passing through *A*, *B*, and *C*; and draw *P A*, *P B*, and *P C*, arcs of great circles. In similar manner, having found *Q*, the pole of the small circle passing through *D*, *E*, and *F*, draw *Q D*, *Q E*, and *Q F*, arcs of great circles.

Demonstration: The small circle that may be passed through *A*, *B*, and *C* is equal to that which may be passed through *D*, *E*, and *F*, for, the sides of the given triangles being arcs that are equal, each to each, the corresponding chords are equal. The plane triangles formed by these chords are, therefore, equal. (BOOK II., PROP. III.) The plane triangles

may, then, be made to coincide, and the small circles will coincide; and they are, therefore, equal.

The small circles being equal, their polar distances are equal (PROP. X., COR. II.); that is, the arcs PA, PB, PC, QD, QE, and QF are equal.

The triangles PAC and QDE are, then, isosceles triangles, and they have the three sides of the one equal to the three sides of the other, each to each. The same is true of the triangles PAB and QDF; and, also, of PBC and QFE.

The corresponding isosceles triangles may be proved equal, by superposition, with respect to the corresponding trihedral angles at the center of the sphere. As the sides of PAC and QDE are respectively equal, the corresponding plane angles at the center are equal. The corresponding trihedral angles are, therefore, equal. (BOOK VIII., PROP. XIV.) The trihedral angles may, then, be made to coincide, and, therefore, the included spherical surfaces, that is, the corresponding isosceles spherical triangles, must coincide; otherwise, points in the surface of the sphere would be unequally distant from the center.

PAC coincides, then, with QDE, and is equal to it. In similar manner, PAB may be proved equal to QDF; and PBC, to QFE.

By equal addition and subtraction (AXIOMS III. and IV.),

$$PAB + PBC - PAC = QDF + QFE - QDE.$$

The first member is equal to ABC, and the second, to DEF; then, ABC and DEF are equivalent. (AXIOM I.)

Conclusion: ABC and DEF being any symmetrical, etc.

Corollary I. Symmetrical spherical pyramids are equivalent.

Scholium. If the poles are within the triangles, the triangles are then equal to the sums of their three respectively equal isosceles triangles.

MISCELLANEOUS EXERCISES.

572. 1. The smallest circle whose plane passes through a given point within a sphere is that one whose plane is perpendicular to the radius through the given point.

2. Two spherical triangles are equal in surface, if their polar triangles have equal perimeters.

3. The radius of a spherical section is one-half the radius of the sphere, if the polar distance of its circumference is one-third of a quadrant.

4. Find the volume of the crust of the earth, 20 miles thick, assuming the earth to be a sphere with a radius of 4,000 miles.

5. What is the volume of the earth's atmosphere, if it extends 50 miles from the surface?

6. If the radius of a sphere is 12 ft., what is the volume of a spherical wedge whose angle is 45°?

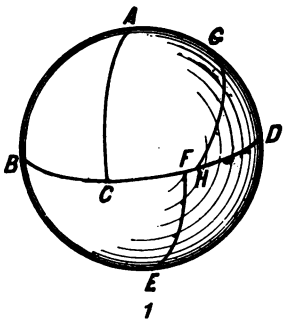
7. The three planes passing through the edges of a trihedral angle perpendicular to the opposite faces intersect in the same line.

8. The altitude of a regular tetrahedron equals the sum of the four perpendiculars from any point within to the four faces.

9. Find the section of a cube by a plane making a regular hexagon.

10. The volume of a right circular cylinder equals the lateral area multiplied by one-half of the radius.

SECTION V.—SPHERICAL TRIANGLES.



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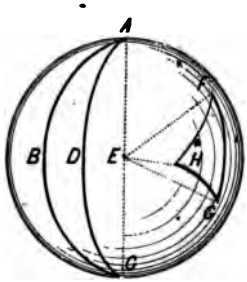

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DEFINITIONS.

573. A spherical triangle may have one, two, or even three right angles, or, obtuse angles.

574. A spherical triangle having a right angle is said to be **rectangular**, as *G H D;* it is **birectangular**, if it has two right angles, as *D E F;* and **trirectangular**, if it has three right angles, as, *A B C*, or *A C D*, in figure *1*.

575. The **spherical excess** of a spherical triangle is the excess of the sum of its angles over two right angles.

576. The *trirectangular triangle* is the **unit** of **spherical surface.** Four units make a hemisphere ; and eight, a sphere.

577. A **lune** is the spherical surface included between two semi-circumferences of great circles ; as, *A B C D*, in figure *2*.

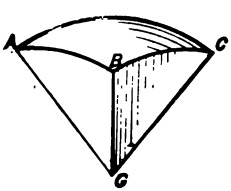
578. A **spherical wedge** is the portion of a sphere bounded by a lune as base, and the planes of its sides ; as, *B D E*.

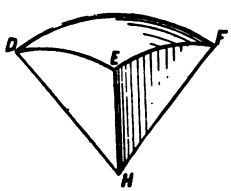
579. A **spherical pyramid** is that which is included between a spherical polygon and the planes of its sides ; as, *E–F H G*.

PROPOSITION XIII.

580. Theorem: ***Spherical triangles on the same or on equal spheres are equal, when they have their parts, taken in the same order, equal, each to each, as follows; either,***

I. Two sides and the included angle; III. Three sides;
II. Two angles and the included side; IV. Three angles; or,
V. In right-angled triangles, the hypotenuse and one side or angle.





Statement: Let *A B C* and *D E F* be any two spherical triangles, fulfilling the conditions in any one of the preceding cases.

Construction: Draw the radii to the vertices of the triangle.

Demonstration: I., II., III., IV., and V. are all proved by reference, in each case, to the corresponding plane and trihedral angles formed at the center by the radii. (BOOK VIII.)

I., II., III., and V. may, also, be proved as the same propositions, with plane triangles, are proved in BOOK II.

Conclusion: As *A B C* and *D E F* are any spherical, etc.

Corollary I. Spherical triangles on the same or on equal spheres are equivalent, when they are symmetrical.

PROPOSITION XIV.

581. Theorem ***Many propositions are equally true with spherical and plane triangles; thus,***

I. In an isosceles triangle, the angles opposite the equal sides are equal.

II. If two angles of a triangle are equal, the sides opposite are equal.

III. If a triangle is equilateral, it is also equiangular.

IV. If a triangle is equiangular, it is also equilateral.

V. The greater side of any triangle has the greater angle opposite.

VI. The greater angle of any triangle has the greater side opposite.

Scholium. The demonstrations are identical in method with the demonstrations of the same propositions with plane triangles, in BOOK II.

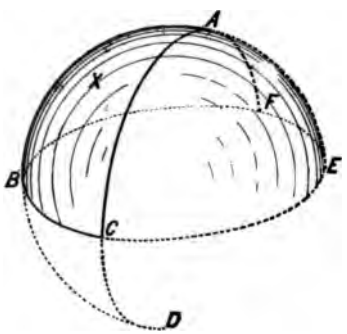
Exercises. 1. Find the volume of a sphere, the diameter being 12.

2. What is the volume of a sphere whose radius is 7?

3. Two spheres have radii, 5 and 6. If the first has a volume equal to 28 cu. in., what is the volume of the second?

PROPOSITION XV.

582. Theorem: ***The area of a spherical triangle is equal to its spherical excess multiplied by a trirectangular triangle.***



Statement: Let *X* be any spherical triangle. Its area is equal to its spherical excess multiplied by the trirectangular triangle of the sphere.

Construction: Complete one of the great circles of which the sides of the triangles are arcs; thus, complete *B C E F*. Produce the other sides to meet the great circle *B C E F*. Beyond the great circle, produce the sides of the angle *A*, so as to complete its lune.

Demonstration: The angle *A* is the vertical angle of two triangles, *B A C* and *F A E*; *B* is the angle of the lune *B A E C*; and *C*, of the lune *C A F B*.

The triangles *B A C* and *F A E* are, however, equal to the lune whose angle is *A*; that is, to *A B D C*, as the triangle *F A E* may

be proved equal to BDC. The three sides of the one may be proved equal, each to each, to the three sides of the other; thus, DA equals CF, as each is a semi-circumference, and, if the common part CA is subtracted, there remains DA, equal to AF. In similar manner, it is proved that BC equals EF, and BD equals AF. The triangles are, then, equal (PROP. XIII.), and the two triangles have, thus, been proved equivalent to the lune whose angle is A.

By a course of reasoning entirely analogous to that in BOOK VI., PROP. I., and PROP. VI. and PROP. XI., it may be proved that the surface of any lune is to the surface of the sphere as the angle of the lune is to four right angles; that is, $L : 8T :: A : 4$, in which L may represent any lune; T, the trirectangular triangle; and A, the angle of the lune.

Any lune equals, therefore, $\frac{8 \times A \times T}{4}$, or $2 \times A \times T$; that is, a trirectangular triangle, multiplied by twice the angle of the lune.

It has been proved, that for each angle of the triangle there is a lune in the hemisphere A–$BCEF$. The sum of these three lunes must, then, be equal to

$$(2A + 2B + 2C) \times T.$$

It is obvious that they exceed the hemisphere, or $4T$, by twice the area of the given triangle; that is,

$$(2H + 2B + 2C) \times T = 4T + 2X.$$

By transposing and dividing, according to the axioms,

$$2X = (2A + 2B + 2C - 4) \times T; \text{ and,}$$

$X = (A + B + C - 2) \times T$; or, X equals the spherical excess multiplied by a trirectangular triangle.

Conclusion: As ABC is any spherical triangle, etc.

Corollary I. The vertical triangles formed on a hemisphere by the intersection of two semi-circumferences of great circles are equivalent to a lune whose angle is the vertical angle of the two triangles.

Corollary II. The area of a lune, $2A\,T$, may be said to be equal to twice the angle of the lune, T, the trirectangular triangle, being represented by 1 as the unit of area

Corollary III. The area of a spherical polygon is equal to its spherical excess multiplied by a trirectangular triangle; that is, to the sum of its angles diminished by two right angles, taken as many times as the polygon has sides less two, multiplied by a trirectangular triangle.

Scholium I. By analogous reasoning, it may be proved that a *spherical wedge* is equal to twice the angle of the lune that forms its base multiplied by the trirectangular pyramid.

Scholium II. A *triangular spherical pyramid* is equal to the spherical excess of its base multiplied by the trirectangular pyramid.

Scholium III. A *spherical pyramid* with any polygon for its base is equal to the spherical excess of that polygon multiplied by the volume of the trirectangular pyramid.

MISCELLANEOUS EXERCISES.

583. 1. The sum of the two arcs of great circles drawn from any point within a spherical triangle to the extremities of any side is less than the sum of the two other sides of the triangle.

2. Draw an arc perpendicular to a given spherical arc, from a given point without it.

3. Bisect an arc of a great circle.

4. Prove that the sides opposite the equal angles of a birectangular triangle are quadrants.

5. Prove that the sum of two angles of a spherical triangle is less than the third angle increased by two right angles.

6. Bisect a spherical angle.

7. Construct a spherical triangle, two sides of the included angle being given.

8. Construct a spherical triangle, two angles and the included side being given.

9. Construct a spherical triangle, the three sides being given.

10. Construct a spherical triangle, the three angles being given.

11. Construct a sphere tangent to three given planes.

12. Construct a sphere tangent to two planes and a given sphere.

13. Construct a sphere tangent to one plane and two spheres.

14. Construct a sphere tangent to three given spheres.

15. Through any given straight line without a sphere pass a plane parallel to the sphere.

16. Prove that the surface of a sphere is equal to the lateral surface of the circumscribing cylinder.

17. Prove that the surface of the sphere is to the entire surface of the circumscribing cylinder, as 2 : 3.

18. Prove that the volume of the sphere is to the volume of the circumscribing cylinder, as 2 : 3.

19. A plane bisecting a dihedral angle is everywhere equally distant from the planes of the given angle.

20. Any point not in the bisecting plane is unequally distant from the planes of the dihedral angle.

21. Find the locus of a point in space equidistant from the extremities of a given line.

22. Find the locus of a point in space equidistant from two parallel planes.

23. Find the locus of a point in space equidistant from two intersecting planes.

24. Find the locus of a point in space equidistant from two parallel planes, also, equidistant from any two points.

25. An oblique straight line makes with its projection on a plane the least angle which it makes with any line in the plane.

26. Find the locus of a point in space equidistant from the three edges of a trihedral angle.

27. Find the locus of a point in space equidistant from the three faces of a trihedral angle.

28. Tetrahedrons with two equal solid angles are proportional to the products of the edges of the equal angles.

29. Similar polyhedrons may be separated into the same number of similar tetrahedrons, similarly placed.

30. In any polyhedron, the number of edges increased by two is equal to the number of vertices increased by the number of faces. (Euler's Theorem.)

31. The sum of the plane angles in the surface of any polyhedron equals four right angles multiplied by the number of vertices, minus two.

32. A sphere may be inscribed in any tetrahedron.

33. A sphere may be circumscribed about any tetrahedron.

34. The arc of a great circle is the shortest distance between any two points on the surface of a sphere.

35. The polar distance of any arc of a great circle is a quadrant.

36. A plane perpendicular to a radius at its outer extremity is tangent to the sphere.

37. A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.

38. Construct a spherical triangle, the poles of the respective sides being given.

39. The intersection of two spheres is the circumference of a circle whose plane is perpendicular to the line joining the centers and whose center is the point of intersection with this line.

40. The sides of a spherical polygon are together less than the circumference of a great circle.

41. The equal sides of a birectangular spherical triangle are quadrants, and the third side is the measure of their included angle.

42. The sides of a trirectangular spherical triangle are quadrants.

43. Erect a perpendicular to the given arc of a great circle from a point on the arc.

44. From a point without an arc of a great circle, let fall a perpendicular to the arc.

45. Find the surface of a sphere, the diameter being 12.

46. Find the surface of a lune, the angle being 45°, the radius being 10.

47. What is the area of a spherical triangle, the angles being 45°, 90°, and 120°, the radius being 5?

48. Find the radius of a sphere, if the surface is 500 square feet.

49. Find the volume of a sphere, if the surface is 1,000 square inches.

50. What is the area of a spherical quadrilateral, the angles being 120°, 130°, 140°, and 150°, the volume of the sphere being 1,000 cubic feet?

51. Find the area of the spherical pentagon, the angles being 90°, 110°, 130°, 150°, and 170°, the diameter being 20 feet.

52. Find the volume of a triangular spherical pyramid, the angles of the base being 60°, 120°, and 240°, and the radius being 12 feet.

53. Find the volume of a spherical wedge, the angle being 72°, the volume of the sphere being 1,728 cubic feet.

54. Find the volume of a spherical segment, the radius being 25 feet, the planes of the bases being 10 feet and 15 feet from the center.

55. Find the entire surface of a spherical cone, the radius of the sphere being 10 feet, and the radius of the base being an arc of 10°.

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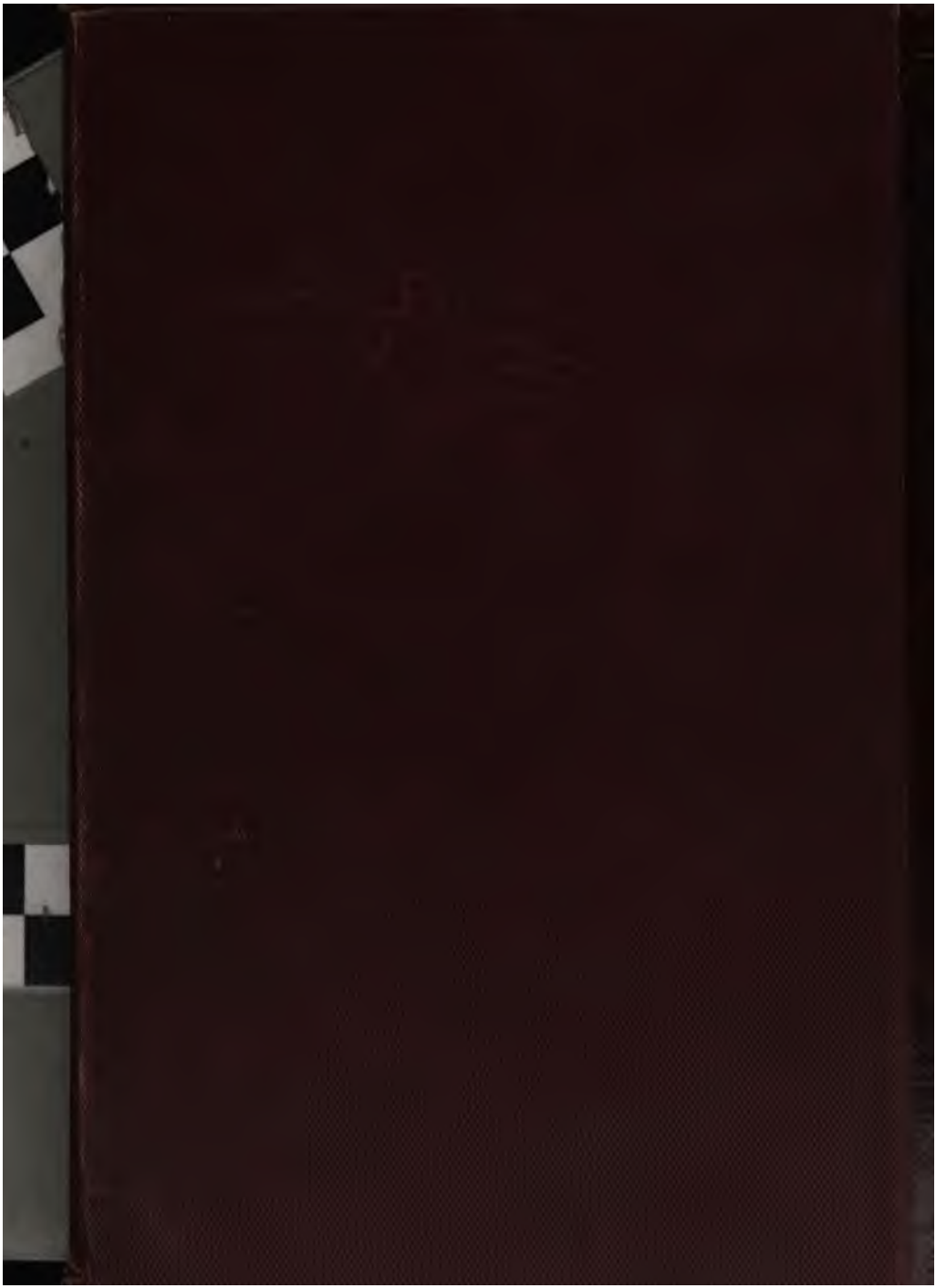
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